Fall 2019

Topology

WVU Mathematics Department

Topology **Ph.D. Entrance Exam** **Fall 2019**

September 10, 2019

#	1	2	3	4	5	total
score						

Directions: Solve all of the following problems. Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1. Write the solution on **one side** of the paper and stay **within the borders**. Anything written outside the borders will not be taken into account. Return your solutions with pages in correct order **arranged according to problem numbers** and together with this cover page.

(1) Prove, or give a counterexample, to each of the following statements.
   (a) Continuous image of a separable space is separable. (Include definition of a separable space.)
   (b) Continuous image of a Lindelöf space is Lindelöf. (Include definition of a Lindelöf space.)
   (c) Continuous image of a completely regular space is completely regular. (Include definition of a completely regular space.)

(2) Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be compact metric spaces and let $f : X \to Y$ and $g : X \to Y$ be continuous. Show that if $f(x) \neq g(x)$ for all $x \in X$, then there exists a $\varepsilon > 0$ such that $\rho(f(x), g(x)) > \varepsilon$.

(3) Let $\{T_a\}_{a\in A}$ be a family of connected subsets of a topological space X. Let $Y := \bigcup_{a\in A} T_a$ and assume that for every $x, y \in Y$ there is a finite sequence $a_0, \ldots, a_n \in A$ such that $x \in T_{a_0}$, $y \in T_{a_n}$, and $T_{a_i} \cap T_{a_{i+1}} \neq \emptyset$ that for every $i < n$. Prove that Y is connected.

(4) Let X be a regular Hausdorff space and A be an infinite subset of X. Prove that there is a sequence $U_1, U_2, \ldots$ of open subsets of X such that $\overline{U_n} \cap \overline{U_m} = \varnothing$ when $m \neq n$ and $U_n \cap A \neq \varnothing$ for each $n \in \mathbb{N}$.
   Hint: Construct the sequence $\langle U_n : n \in \mathbb{N}\rangle$ by induction so that $A \setminus \overline{U_n}$ is infinite for every n.

(5) Let $X = \mathbb{N}^{\mathbb{N}}$ have the product topology assuming that $\mathbb{N}$ has discrete topology. Let $Y \subseteq X$ be a compact subspace of X.
   (a) Prove that for each $n \in \mathbb{N}$, there is finite $F_n \subseteq \mathbb{N}$ such that $Y \subseteq \prod_{n\in\mathbb{N}} F_n$.
   (b) Show that X is not a countable union of compact subsets of X.
   Hint: In the proof of part (b) use part (a) and a diagonal argument to show that for every sequence $\langle Y_n : n \in \mathbb{N}\rangle$ of compact subsets of X there is an $x \in X$ that belongs to no Y_n.

page # of problem # | name: