M.S. Advanced and Ph.D. Entrance Exams | Mathematics

Fall 2019

Real Analysis

WVU Mathematics Department

Entrance Exam, Real Analysis

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Solve exactly 6 out of the 8 problems.

1. (a) Show whether $f(x) = x^2$ is absolutely continuous on $[0, 1]$.

   (b) Show whether $f(x) = x^2$ is absolutely continuous on $[0, \infty)$.

2. Consider

$$f_n(x) = \begin{cases} n^2 x & 0 \le x < \frac{1}{2n}, \\ -n^2 x + n & \frac{1}{2n} \le x < \frac{1}{n}, \\ 0 & \frac{1}{n} \le x \le 1, \end{cases} \quad n = 1, 2, \ldots.$$

   (a) Show whether the sequence $\{f_n\}$ converges pointwise to zero on $[0, 1]$.

   (b) Show whether the sequence $\{f_n\}$ converges almost everywhere to zero on $[0, 1]$.

   (c) Show whether the sequence $\{f_n\}$ converges uniformly to zero on $[0, 1]$.

3. Suppose that $f(x)$ is a uniformly continuous and Lebesgue integrable function on $\mathbf{R}$. Show that

$$\lim_{|x| \to \infty} f(x) = 0.$$

4. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions on $[0, 1]$ and assume $\lim_{n \to \infty} u_n(x) = 0$ a.e. on $[0, 1]$, and also $\|u_n\|_{L^3[0,1]} \le 1$ for all n. Prove that

$$\lim_{n \to \infty} \|u_n\|_{L^1[0,1]} = 0.$$

5. Answer the following questions and prove your conclusion.

   - If A is a closed subset of [0,1] and $A \ne [0, 1]$. Is it possible that $mA = 1$?
   - If B is an open subset of [0,1] and dense in $[0, 1]$. Is it possible that $mB < 1$?

6. Let $f(x)$ be defined by $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}$.

   - Prove than $f(x)$ is continuous at any $x \in (-\infty, \infty)$;
   - Is $f(x)$ uniformly continuous in $(-\infty, \infty)$? Prove your conclusion.

7. If $f(x)$ is integrable over $[0, 1]$, then

$$\lim_{\lambda \to \infty} \int_0^1 f(x) cos(\lambda x) dx = 0.$$

8. Let $f(x)$ be a function of bounded total variation defined on [0,1]. Prove that the set of all discontinuity points of $f(x)$ in [0,1] is countable.