Fall 2019

Differential Equations

WVU Mathematics Department

ODE ENTRANCE EXAM FALL 2019

Show all work. Explain and justify your answers. All problems carry equal weight. Do all six problems.

Name ______________________________ Score ________

1. a) Explain (without proofs) the method of successive approximations. Formulate a theorem of existence and uniqueness for first order nonlinear systems of ODE's that is based on it.

b) Show that the conditions of the theorem in a) are satisfied by the initial value problem

$$\frac{dy_1}{dt} = \frac{2y_1 + y_2}{y_2 - 1}, \; \frac{dy_2}{dt} = y_1 y_2 + t, \; y_1(0) = 1, \; y_2(0) = 3. \tag{1}$$

c) Estimate an interval of existence (validity) for the solution to (1).

2. a) Determine the solution of the initial value problem

$$\frac{dy}{dt} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} y - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; y(0) = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}. \tag{2}$$

Does $\lim_{t \to -\infty} y(t)$ exist?

b) For which values of $\alpha, \beta, \gamma, \delta$ does the initial value problem

$$\frac{dy}{dt} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} y - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; y(0) = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}, \tag{3}$$

possess solutions that are bounded on $(-\infty, \infty)$?

3. Consider the first order scalar autonomous differential equation

$$\frac{dy}{dt} = (y^3 - 1)(y^2 - 4). \tag{4}$$

a) Determine all its critical points.

b) Sketch the graphs of solutions of (4) in a (t, y) Cartesian plane and explain how do the integral curves $y(t)$ depend on the initial values $y_0 := y(0)$.

c) Let $P_n(y)$ be a scalar polynomial in y of degree $n \in \mathbb{N}$. Consider the first order scalar autonomous differential equation

$$\frac{dy}{dt} = P_n(y). \tag{5}$$

Prove that (5) must have unbounded solutions on $(-\infty, \infty)$.

4. Let $\omega > 0$ be a positive constant, and let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Consider the second order differential equation

$$y'' + [\omega^2 + f(t)]y = 0. \tag{6}$$

a) Assuming

$$\int_0^\infty |f(t)| dt < \infty, \tag{7}$$

show that the zero solution of (6) is uniformly stable.

b) Show that the result in part b) does not necessarily hold if condition (7) is dropped.

5. Consider the initial value problem

$$\frac{dx}{dt} = t^2 + x^2, \qquad x(0) = 0 \tag{8}$$

a) Show that (8) has a unique solution defined on an interval for t containing $(-\sqrt{2}/2, \sqrt{2}/2)$.

b) Show that the solution of (8) is not defined for all $t \in (-\infty, \infty)$.

6. Consider the system

$$\begin{cases} x' = y \\ y' = -2x^3 - x. \end{cases} \tag{9}$$

a) Show that for any $(t_0, x_0, y_0) \in \mathbb{R}^3$, the initial value problem $x(t_0) = x_0$, $y(t_0) = y_0$ has a unique solution $(x(t), y(t))$ defined for all $t \in (-\infty, \infty)$.

b) Show that all solutions of the system are periodic.