Fall 2017

Differential Equations

WVU Mathematics Department

PhD Entrance Exam, Differential Equations; Fall 2017

INSTRUCTIONS (Please read carefully!): Solve any 6 (SIX) PROBLEMS, and clearly indicate which ones are to be graded by encircling the problem numbers on these exam pages! ONLY THE FIRST 6 (SIX) ENCIRCLED PROBLEMS WILL BE GRADED!

Below $\dot{f}$ represents the derivative of the function f with respect to the real variable t. $C(\mathbb{R}^d)$ denotes the set of continuous functions on $\mathbb{R}^d$, while $C^k(\mathbb{R}^d)$ denotes the set of functions on $\mathbb{R}^d$ with continuous derivatives of up to and including k^{th} order.

1. (a) Solve the initial value problem

$$\dot{y} = Ay - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

expressing your solution in finite terms with $A = TJT^{-1}$, where

$$T = 2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Determine $\lim_{t \to \infty} y(t)$.

2. Given the autonomous differential system

$$\dot{y} = f(y), \ f \in C^1(\mathbb{R}^n), \tag{1}$$

let $y(t)$ be a continuous $n \times 1$ column vector solution of (1) on the interval (a, b). Prove the following:

(i) If $b = \infty$ and $\lim_{t \to \infty} y(t) = L$ for some constant vector L, then L is a critical point of (1).

(ii) Assume the derivative $\dot{y}$ is not identically zero on (a, b) and that $\lim_{t \to b^+} y(t) = L$, where L is a critical point of (1). Prove that $b = \infty$.

(iii) $y(t+\omega)$ is also a vector solution of (1) on $(a+\omega, b+\omega)$.

3. (a) Estimate an interval of existence for the initial value problem

$$\dot{y}_1 = \frac{y_2}{1-y_1}, \quad \dot{y}_2 = \frac{y_1}{1+y_2}, \qquad y_1(0) = 2, \quad y_2(0) = 1. \tag{2}$$

(b) Set up an equivalent integral equation for (2) and determine the first two successive approximations for the solution.

4. Provide examples for the following:

(a) An example of an initial value problem $\dot{y} = f(y)$, $y(0) = 0$ with $f \in C(\mathbb{R})$ such that the initial value problem does not have a unique solution.

(b) An example of an initial value problem $\dot{y} = f(y)$, $y(0) = y_0$ with $f \in C(\mathbb{R})$, with solutions that do not exist on the whole $\mathbb{R}$

(c) An example of a scalar linear homogeneous singular differential equation such that all solutions exist on the whole $\mathbb{R}$.

5. Discuss the stability of the zero solution for

$$\dot{x} = x(\cos y - y) - y(1+t^2)^{-1}, \quad \dot{y} = -2\sin x + y - 4x\sin y + 2x(1+t)^{-2}.$$

6. Consider the differential system

$$\dot{\mathbf{x}} = -A(t)\mathbf{x} - |\mathbf{x}|\mathbf{x}, \quad t \geq 0, \tag{3}$$

where $A : [0,\infty) \to \mathbb{R}^{n\times n}$ is a matrix-valued continuous function, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

(a) Assume that there exists a constant $0 < B < \infty$ such that all the entries of the matrix $A(t)$ satisfy

$$|A_{ij}(t)| \leq B \text{ for all } t \in [0,\infty) \text{ and all } 1 \leq i,\ j \leq n.$$

Prove that for any $\mathbf{x}_0 \in \mathbb{R}^n$ the initial value problem given by (3) and $\mathbf{x}(0) = \mathbf{x}_0$ has a unique solution defined on $[0,\infty)$.

(b) Assume further that there exists a continuous function $\lambda : [0,\infty) \to (0,\infty)$ such that

$$\int_0^\infty \lambda(t)dt = \infty$$

and

$$\mathbf{v}^T A(t)\mathbf{v} \geq \lambda(t)|\mathbf{v}|^2 \text{ for all } t \in [0,\infty) \text{ and all column vectors } \mathbf{v} \in \mathbb{R}^n,$$

where $\mathbf{v}^T$ and $|\mathbf{v}|$ denote the transpose and the Euclidean norm of $\mathbf{v}$, respectively. Prove that the zero solution of (3) is globally asymptotically stable.

7. Consider a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the scalar differential equation

$$\dot{x} = f(t, x) \tag{4}$$

has the following property: for all $t_0,\ x_0 \in \mathbb{R}$, the initial value problem (4) together with $x(t_0) = x_0$ has at least one solution defined on the whole $\mathbb{R}$.

Prove the following statement: for any $c_1,\ c_2 \in \mathbb{R}$ and any solutions $x_1,\ x_2$ of (4) the function $c_1x_1 + c_2x_2$ is also a solution of (4) if and only if there exists a continuous function $a : \mathbb{R} \to \mathbb{R}$ such that $f(t, x) = a(t)x$ for all $t,\ x \in \mathbb{R}$.

8. Consider the system of scalar differential equations

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y),$$

where f, g are continuous real-valued functions defined on $\mathbb{R}^2$ for which there exists a *nonnegative* real-valued function h such that

$$x[2f(x,y)+g(x,y)]+y[f(x,y)+2g(x,y)] = h(x,y)(1-x^2-y^2) \text{ for all } x,\ y \in \mathbb{R}.$$

Prove that the system has periodic solutions.

Odes-phd entrance exam spring 2018

Name (Print).__________________

Show all Work. Draw and Explain. All problems carry the same weight.
Do 6 out of the 8 problems and mark the 6 problems you want to be checked.

1. (a) Find the general solution and discuss the stability of the system

$$\dot{x} = Ax = \begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} x.$$

   (b) Discuss the stability of the system for $t \geq 1$.

$$\dot{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} -x_1 + t^{-2}x_1 + x_2 - 2x_3 - 2e^{-t}x_3 + e^{t^2} \\ t^{-2}x_2 - 2x_3 + t \\ e^{-2t}x_1 - x_3 + 1 \end{bmatrix}.$$

2. Consider

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x^3 = 0.$$

   (a) Show whether the zero solution is stable, asymptotically stable, or unstable.

   (b) Show whether the equation has a limit cycle.

3. Discuss the stability of critical points.

   (a)

$$\begin{aligned} \dot{x} &= -y + xy^2, \\ \dot{y} &= x - 2x^2 y. \end{aligned}$$

   (b)

$$\begin{aligned} \dot{x} &= y^3 + x^2 y, \\ \dot{y} &= x^3 - 2xy^2. \end{aligned}$$

4. (a) Discuss the stability of critical points.

$$\begin{aligned} \dot{x} &= y(1 + x^3), \\ \dot{y} &= x(y + 1). \end{aligned} \tag{1}$$

   (b) Sketch the phase diagram of the system (1).

5. (a) Determine the vector solution to the initial value problem

$$y' = \frac{dy}{dt} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} y - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; y(0) = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \eta \end{bmatrix} \tag{2}$$

   as a finite sum of vectors.

   (b) Determine all values of the parameters $\alpha, \beta, \gamma, \eta \in \mathbb{R}$ that render the solution in a) bounded on $[0, \infty)$. Are there any parameters $\alpha, \beta, \gamma, \eta \in \mathbb{R}$ that render a solution to 2 periodic on $(-\infty, \infty)$? If yes determine all of them.

6. (a) Formulate and prove the Gronwal inequality. Explain the importance of the Gronwal inequality. What is it good for?

   b) Under what conditions on $y(t)$ does the inequality

$$y(t) \leq \int_0^t [sin^2 y(u)]du$$

   imply that $y(t) \equiv 0$ for $t \in [0,1]$? Justify your answer.

7. Given a differential system

$$y' = f(t,y).$$

   (a) Formulate an existence and uniqueness theorem for the above nonlinear system. Explain the method of successive approximations.

   (b) Prove that all solutions to

$$y_1' = cos(ty_1y_2),\ y_2' = sin(y_1^2 + y_2^3 + t) \tag{3}$$

   exist on $(-\infty, \infty)$.

8. Given the scalar initial value problem

$$2x'' + 6x^5 = 0,\ x(0) = \alpha,\ x'(0) = \beta,\ \alpha, \beta \in \mathbb{R}. \tag{4}$$

   (a) Show that if $x(t)$ is a solution then

$$[x'(t)]^2 + x^6(t) = \beta^2 + \alpha^6. \tag{5}$$

   (b) Show that (4) possesses a bounded solution with a bounded derivative on $(-\infty, \infty)$.

   (c) It is given further that $\alpha > 0, \beta > 0$. Show that the solution $x(t)$ attains a (relative) local maximum at a time $t_{max} > 0$. Determine precisely the values of $x(t_{max})$, $x'(t_{max})$, $x''(t_{max})$ in terms of α, β. Show that there exists a time $t_{min} > t_{max} > 0$ where $x(t)$ attains a local minimum. Determine precisely the values of $x(t_{min})$, $x'(t_{min})$, $x''(t_{min})$.

   (d) Sketch the orbit of this solution for $t_{max} \leq t \leq t_{min}$ in the phase space. Also sketch in a (t,x) plane the graph of $x(t)$ on the interval $t_{max} \leq t \leq t_{min}$. What is the sign of $x'(t)$ on this interval? Conclude that there is a one to one correspondence between the values of t and $x(t)$ on $t_{max} \leq t \leq t_{min}$.

Differential Equations Entrance Exam, 2018f. NAME:
Solve 6 problems. Indicate 6 problems of your choice.

1. (a) Assume that the scalar function $w(t)$ and it derivative $w'(t)$ are continuous on $[a, \infty)$ and that

$$lim_{t\to\infty} w(t) = L_1,\ lim_{t\to\infty} w'(t) = L_2. \tag{1}$$

Prove that $L_2 = 0$. Hint: you may use a relation like $w(t) = w(a) + \int_a^t w'(s)ds$ and argue by contradiction.

(b) Given the autonomous differential system

$$\frac{dy(t)}{dt} = f(y),\ f(y) \in C^1(\mathbb{R}^n). \tag{2}$$

Let $y(t)$ be a continuous n by 1 column vector solution of (2) on (a, b). Prove the following. If $lim_{t\to\infty} y(t) = L$, L being a constant vector, then L is a critical point of (2). Hint:use i)

(c) Assume $\frac{dy(t)}{dt}$ is not identically zero and that $lim_{t\to b^+} y(t) = L$, where L is a critical point of (2). Then $b = \infty$.

2. (a) Given the differential system

$$\frac{dy(t)}{dt} = f(t, y). \tag{3}$$

Formulate an existence theorem for solutions of initial value problems to (3). Discuss and explain the method of successive approximations. What are its goals? What assumptions on (3) are needed to make the method of successive approximations guarantee a unique solution. How is the Gronwall lemma used in this method?

(b) Estimate an interval of existence for the initial value problem

$$y_1' = y_1 y_2,\ y_2' = \frac{y_2}{10 - y_1},\ y_1(0) = 1,\ y_2(0) = 1. \tag{4}$$

(c) Does

$$y_1' = y_1 y_2,\ y_2' = \frac{y_2}{10 - y_1}$$

(4) possess solutions on $(-\infty, \infty)$? If yes which are they?

3. Given the differential system

$$y' = Jy, J = \begin{bmatrix} 2i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}. \tag{5}$$

(I) Determine a fundamental set of vector solutions on $(-\infty, \infty)$.

(II) Determine all vector solutions of (5):

(a) That are bounded for $t > 0$.

(b) That are bounded on $t < 0$.

(c) That are periodic non constant solutions and determine the smallest positive period.

4. (i) Formulate and prove the Gronwall inequality.

   (ii) Show that if $f(t)$ is a non negative continuous function on $[0, 1]$ and if

$$f(t) \leq \int_0^t 3[|sin(f(s))|]ds, \; t \in [0, 1],$$

   then $f(t) \equiv 0$ on $[0, 1]$.

5. Consider

$$x'' + (x^2 + (x')^2 - 1)x' + x^3 = 0,$$

   where $x' = \frac{dx}{dt}$ and so forth.

   (a) Show whether the zero solution is stable, asymptotically stable, or unstable.

   (b) Show whether the equation has a limit cycle.

6. (a) Discuss the stability of critical points.

$$\begin{aligned} x' &= y + xy, \\ y' &= xy^3 + x. \end{aligned} \tag{6}$$

   (b) Sketch the phase diagram of the system (6). Indicate the increasing direction of time.

   (c) Among the following lines which are invariant? If none are invariant, state so.

   (i) $y = 0$, (ii) $x = -1$, (iii) $x = 0$, (iv) $y = -1$.

7. (a) Verify that $y = t$ $(t > 0)$ is a solution for the second order differential equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0.$$

   (b) Find the general solution for

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \; t > 0.$$

8. Consider

$$x' = f(x), \tag{7}$$

   where $x \in R^n$, f is continuously differentiable, $f(0) = 0$, and $x = 0$ is an isolated critical point. Suppose that the linearized system is given by

$$x' = Ax, \tag{8}$$

   where

$$A = \frac{\partial f}{\partial x}(0)$$

   is the Jacobian of f evaluated at 0. Suppose the zero solution of (8) is stable. Either prove that the zero solution of (7) is stable or show a counter example.

Differential Equations Entrance Exam, 2019S. NAME:
Solve 6 problems. Indicate 6 problems of your choice.

1. Assume that A is an n by n constant upper triangular matrix
$$A = [a_{j,k}], \quad a_{k,k} = k\sqrt{-1},\ k = 1, 2, \cdots, n. \tag{1}$$
Consider on $[0, \infty)$ the vector differential system
$$y' = Ay. \tag{2}$$
Show that the zero solution $y(t) \equiv \overrightarrow{0}$ is stable but not asymptotically stable.

2. Given $y \in \mathbb{R}^2$, $\nu > 0$, $\sigma < 0$ and that
$$\frac{dy(t)}{dt} = \begin{bmatrix} \sigma & \nu \\ -\nu & \sigma \end{bmatrix} y(t). \tag{3}$$
Determine a fundamental matrix solution to (3) that has real valued entries. Sketch the phase portrait of (3) as $t \to \infty$. Carefully explain the direction of arrows along the orbits.

3. (a) Discuss the stability of the system. Here, $x = x(t)$, $\dot{x} = \frac{dx}{dt}$ etc, and $t \in R$.
$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} x.$$
The characteristic polynomial of the above coefficient matrix A is $\lambda^2(\lambda + 1)$.

   • In the next two questions the initial data $x_0 \in R^3$ are given at $t = 0$, i.e., $x(0) = x_0$.

   (b) Find the set of all initial data for which the solution is bounded for all t. Show that it forms a subspace of R^3.

   (c) Find the set of all initial data for which the solution approaches zero as t approaches infinity. Show that it forms a subspace of R^3.

4. Consider a system in R^2. Here, $x = x(t)$, $\dot{x} = \frac{dx}{dt}$ etc, and $t \in R$.
$$\begin{aligned} \dot{x} &= y + x^3 y, \\ \dot{y} &= x + xy. \end{aligned}$$
(a) Find the critical points and discuss their stability.

   (b) Along what curves (or lines) $\frac{dy}{dx}$ is zero or infinite? Draw these curves in the xy-plane.

   (c) Sketch the phase diagram. Be sure to put "Arrows" in orbits (or paths) to indicate the direction of time.

5. Consider systems in R^2. Discuss the stability of critical points. Here, $x = x(t)$, $\dot{x} = \frac{dx}{dt}$ etc, and $t \in [0, \infty)$.

   (a)
$$\begin{aligned} \dot{x} &= 4x^2 - y^2, \\ \dot{y} &= -2x + xy - 4. \end{aligned}$$
(b)
$$\begin{aligned} \dot{x} &= 2x^2 y + y^3, \\ \dot{y} &= -3xy^2 + x^3. \end{aligned}$$

6. Consider the differential equation

$$\ddot{x} = x^3 - x, \quad x = x(t) \in R, \ \ddot{x} = \frac{d^2x}{dt^2}, \ t \in R.$$

(a) Take $\dot{x}$ as the vertical axis and x as the horizontal axis to draw the phase diagram.

(b) Suppose $x(0) = -1$ and $\dot{x}(0) = 0$. Find the solution. Draw the orbit of the solution in the phase diagram.

(c) Determine if there exists a solution satisfying $x(0) = 1$ and $x(1) = 1$.

(d) Determine if there exists a bounded solution satisfying $x(0) = -1$ and $x(1) = 1$.

7. Given that: i)

$$A(t) = [a_{j,k}(t)], \quad a_{j,k}(t) \in C(-\infty, \infty), \ j, k = 1, 2, \cdots, n. \ t, t_0 \in (-\infty, \infty), \tag{4}$$

ii) Exists $\omega > 0$ such that

$$A(t + \omega) \equiv A(t), \ \omega > 0. \tag{5}$$

Let $M(t)$ be a square n by n fundamental matrix solution of

$$\frac{dM(t)}{dt} = A(t)M(t). \tag{6}$$

Show that:

(a) $M(t + \omega)$ is also a fundamental matrix solution of (6).

(b) Show that the matrix

$$C := M^{-1}(t)M(t + \omega)$$

exists is invertible and actually is a constant independent of t.

(c) It is known that there exists a constant matrix R such that

$$C = exp(\omega R). \tag{7}$$

Prove that

$$P(t) := M(t)exp(-tR)$$

is periodic with period ω and that the transformation $M(t) = P(t)Z(t)$ takes the differential system (6) into the system

$$\frac{dZ(t)}{dt} = RZ(t). \tag{8}$$

(d) Determine the matrices C and $P(t)$ with

$$\omega = 2\pi, \ A(t) = \begin{bmatrix} cos(t) & 1 \\ 0 & cos(t) \end{bmatrix}, \ M(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Can you find the matrix R as well?

8. Given

$$y' = f(t, y, p), \quad y(t_0) = \eta, \quad p \in \mathbb{R}^1, y, \eta, f \in \mathbb{R}^n. \tag{9}$$

Assume: i) $f(t, y, p)$ is a continuous vector function in the set of points (t, y)

$$REC.BOX := \{t \in I, \quad |y - \eta| \le b\} \tag{10}$$

, I an interval

$$I = \{t | \, |t - t_0| \le \delta\} \tag{11}$$

and

$$|f(t, y)| \le M.$$

ii) Assume the entries of the Jacobian matrix

$$JM := (\frac{\partial f_i}{\partial y_j}) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \cdot \\ \cdot \\ \cdot \\ \nabla f_n \end{bmatrix}$$

to be continuous in the $REC.BOX$.

iii) Assume that there exists a solution $y = \phi(t, t_0, \eta)$ to the initial value problem (9) on the interval

$$I = \{t | \, |t - t_0| \le q\}, \quad q = Minimum\, \{\delta, \frac{b}{M}\}.$$

Prove that the solution $y = \phi(t, t_0, \eta)$ is unique.