Fall 2017

Real Analysis

WVU Mathematics Department

Entrance Exam, Real Analysis

September 1, 2017

Solve exactly 6 out of the 8 problems

1. Prove by definition (in $\epsilon - \delta$ language) that $f(x) = \sqrt{1+x^2}$ is uniformly continuous in (0, 1). Is $f(x)$ uniformly continuous in $(1, \infty)$? Prove your conclusion.

2. Let $f_n(x) = \dfrac{nx}{n+x}$.

   (a) Find $f(x) = \lim_{n\to\infty} f_n(x)$;

   (b) Does f_n converge to f uniformly $(0, 1)$? Prove your conclusion.

   (c) Does f_n converge to f uniformly $(1, \infty)$? Prove your conclusion.

3. If $\overline{\lim_{n\to\infty}} s_n \geq 0$, which of the following statements are true? Explain your conclusion.

   (a) $\exists N > 0$ such that $\forall n > N$, $s_n \geq 0$.

   (b) $\forall N > 0$, $\exists n > N$ such that $s_n \geq 0$.

   (c) $\exists \epsilon > 0$ such that $\forall N > 0$, $\exists n > N$ with $s_n \geq 0$.

   (d) $\overline{\lim_{n\to\infty}} s_n^2 \leq (\overline{\lim_{n\to\infty}} s_n)^2$.

4. Let $g(x)$ be defined as follows

$$g(x) = \int_{-\infty}^{\infty} \frac{\cos(xy^3)}{1+y^2} dy.$$

   (a) Prove $g(x)$ is continuous in $(-\infty, \infty)$.

   (b) Is $g(x)$ uniformly continuous in $(-\infty, \infty)$? absolutely continuous in $(-\infty, \infty)$? Prove your conclusion.

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying $\sum_{n=1}^{\infty} |a_n| < \infty$. Define f by $f(x) = \sum_{n=1}^{\infty} a_n x^n$.

   (a) Show that f is a function of bounded variation on $x \in [-1, 1]$.

   (b) If $a_1 \neq 0$ and $a_n = 0$ for $n \geq 2$, is f absolutely continuous on $x \in \mathbf{R}$. Explain your answer.

   (c) If $a_1 \neq 0$, $a_2 \neq 0$, and $a_n = 0$ for $n \geq 3$, is f absolutely continuous on $x \in \mathbf{R}$. Explain your answer.

6. For $1 \le p < \infty$ define

$$\|f\|_p = (\int_E |f|^p)^{1/p}$$

and denote $L^p(E)$ to be the set of functions f for which $\int_E |f|^p < \infty$.

Either prove the statement or show a counter example.

(a) If $E = [0,1]$, then there is a constant $c > 0$ for which

$$\|f\|_1 \le c\|f\|_2 \text{ for all } f \in L^2(E).$$

(b) Suppose $E = [1, \infty)$. If f is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$.

7. Suppose that $f(x)$ is a uniformly continuous and Lebesgue integrable function on $\mathbf{R}$. Show that

$$\lim_{|x|\to\infty} f(x) = 0.$$

8. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions on $[0,1]$ and assume $\lim_{n\to\infty} u_n(x) = 0$ a.e. on $[0,1]$, and also $\|u_n\|_{L^2[0,1]} \le 1$ for all n. Prove that

$$\lim_{n\to\infty} \|u_n\|_{L^1[0,1]} = 0.$$

Entrance Exam, Real Analysis

April 20, 2018

Solve exactly 6 out of the 8 problems below.

1. Let $f(x)$ be continuous in (0,1) and $f(x) > 0$ in (0,1).
   - Prove by definition that $\frac{1}{f(x)}$ is continuous in (0,1).
   - If $f(x)$ is also uniformly continuous in (0,1), is $\frac{1}{f(x)}$ also uniformly continuous? Prove your conclusion.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f(0) = 0$, $f'(0) > 0$ and $f''(x) \geq f(x)$ for all $x \geq 0$. Prove that $f(x) > 0$ for all $x > 0$.

3. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a twice differentiable, nonconstant convex function (so that $\varphi'' \geq 0$ on $\mathbb{R}$). Prove that

$$\text{either } \lim_{x\to-\infty} \varphi(x) = \infty \text{ or } \lim_{x\to\infty} \varphi(x) = \infty\,.$$

4. If $\overline{\lim_{n\to\infty}} s_n \geq 0$, which of the following statements are true? Explain your conclusion.
   (a) $\exists N > 0$ such that $\forall n > N$ we have $s_n \geq 0$.
   (b) $\forall N > 0$, $\exists n > N$ such that $s_n \geq 0$.
   (c) $\exists \epsilon > 0$ such that $\forall N > 0$, $\exists n > N$ such that $s_n \geq -\epsilon$.
   (d) $\forall \epsilon > 0$, $\exists N > 0$, such that $\forall n > N$ we have $s_n \geq -\epsilon$.

5. Let $A \subset \mathbb{R}$ with Lebesgue measure $m(A) = 1$. Show that there exists a measurable subset $B \subset A$ such that $m(B) = \frac{1}{2}$.

6. Let f be a monotone increasing function on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. Let $E = \{f(x)\,;\, x \in [0, 1]\}$. If $m(E) = 1$ (where $m(E)$ is the Lebesgue measure of set E), is it true that f must be continuous on [0,1]? Prove your conclusion.

7. Let $f \in L^\infty(\mathbb{R})$ and $g(x)$ be defined as follows

$$g(x) = \int_{-\infty}^{\infty} \frac{f(x-y)}{1+y^2} dy.$$

   - Prove $g(x)$ is continuous in $(-\infty, \infty)$.
   - Is $g(x)$ absolutely continuous in $(-\infty, \infty)$? Prove your conclusion.

8. Evaluate (with justification)

$$\lim_{n\to\infty} \int_0^n \frac{\cos(x/n)}{x^2 + \cos(x/n)} dx.$$

Analysis Entrance Exam, Fall 2018

Solve any 6 (six) problems and clearly mark which six should be graded by encircling the problem numbers on this sheet. *Only the marked problems will be graded!* If more than six problems are marked, only the first six marked problems will be graded.

1. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers such that $2a_{n+1}^2 - a_n^2 = 1$ for all integers $n \geq 0$. Study the convergence/divergence and find the limit of $(a_n)_{n\geq 0}$ whenever it exists.

2. Let (X, Σ, μ) be a measure space and $\mu : \Sigma \to [0, \infty]$ a finitely additive set function, that is, for any integer $n \geq 1$ one has $\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$ for all pairwise disjoint $A_i \in \Sigma$, $i = 1, ..., n$. If $\lim_{k\to\infty} \mu(E_k) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$ for every increasing sequence of measurable sets $\{E_k\}_{k\geq 1}$ (that is, $E_k \subset E_{k+1}$ for all $k \geq 1$), then prove that μ is a *positive measure*, that is, $\mu(\emptyset) = 0$ and $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ for all pairwise disjoint $\{A_n\}_{n\geq 1} \subset \Sigma$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable (everywhere) function. Prove that the following statements are equivalent (you are *not* allowed to use any fact you may know about convex functions):
(1) $f''(x) \geq 0$ for all $x \in \mathbb{R}$.
(2) $f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$ for all $x,\ y \in \mathbb{R}$ and all $\lambda,\ \mu \in [0, 1]$ such that $\lambda + \mu = 1$.

4. Assume $f,\ g : \mathbb{R} \to \mathbb{R}$ are both continuous at $x = 0$ and $g(0) \neq 0$.
(a) Prove that f/g is continuous at $x = 0$.
(b) If, in addition, f and g are differentiable at $x = 0$, then prove that f/g is differentiable at $x = 0$ and compute $(f/g)'(0)$ in terms of $f(0),\ f'(0),\ g(0),\ g'(0)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}(1 + |\log x|)}, & \text{if} \quad x > 0, \\[2ex] 0, & \text{if} \quad x \leqslant 0. \end{cases}$$

Prove that $f \in L^p(\mathbb{R})$ if and only if $p = 2$.

6. (a) Prove that the product of two bounded (above and below) and uniformly continuous functions on the real line is uniformly continuous.
(b) Prove that the product of two uniformly continuous functions on the real line is not necessarily uniformly continuous if only one of these functions is assumed bounded.

7. Show that $\displaystyle\lim_{k\to\infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k dx = n!$.

8. Let $f : (-1, 1) \to \mathbb{R}$ be twice differentiable with f'' continuous and such that $f(0) = f'(0) = 0$, $f''(0) = 1$. Prove that there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (-\delta, 0) \cup (0, \delta)$.

Analysis Entrance Exam, Spring 2019

Solve any 6 (six) problems and clearly mark which six should be graded by encircling the problem numbers on this sheet. *Only the marked problems will be graded!* If more than six problems are marked, only the first six marked problems will be graded.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with f' continuous and such that $f(0) = 0$ and $f'(0) = 1$.
   (a) Show that there exist $\delta > 0$ and $a < 0 < b$ such that the restriction $f : [-\delta, \delta] \to [a, b]$ is invertible.
   (b) Denote by g the inverse of the restriction of f to $[-\delta, \delta]$. Prove that g is differentiable at 0 and $g'(0) = 1$.

2. For the questions (a), (b), (c) below, the function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be nondecreasing ($f(x) \leq f(y)$ if $x \leq y$) and bounded above.
   (a) Show that $\lim\limits_{x\to\infty} f(x)$ exists and is a real number.
   (b) If f is also continuous and $f(x) = 0$ for all $x \leq 0$, prove that f is uniformly continuous.
   (c) If f is also differentiable, show that there exists a sequence $\{x_n\}_n$ such that $\lim\limits_{n\to\infty} x_n = \infty$ and $\lim\limits_{n\to\infty} f'(x_n) = 0$.

3. Let $\{a_n\}_n$ be a sequence of real numbers such that $0 \leq n^2(a_{n+1} - a_n) \leq 1$ for all integers $n \geq 1$.
   (a) Prove that $\{a_n\}_n$ is convergent.
   (b) Does the conclusion from (a) remain valid if n^2 is replaced by n in the inequalities above?

4. Let $f : [0, 1] \to [0, 1]$ be continuous and such that $f(x) > 0$ for all $x \in [0, 1]$. Prove that $1/f$ is Riemann integrable on $[0, 1]$ and

$$\int_0^1 \frac{dx}{f(x)} \geq \frac{1}{\int_0^1 f(x)dx}.$$

   Find all such functions f for which equality holds.

5. Define

$$f(x) = \begin{cases} x^{1+\epsilon} \sin(\frac{1}{x}), & \text{for } x \in (0, 1], \\ 0, & \text{for } x = 0. \end{cases}$$

   Is f Lipschitz continuous, absolutely continuous or/and of bounded variation on $[0, 1]$ for all $\epsilon > 0$?

6. Let $f : [\alpha, \beta] \to \mathbb{R}$ absolutely continuous. Prove that if $A \subseteq [\alpha, \beta]$ is Lebesgue measurable and $\mu(A) = 0$ then $\mu(f(A)) = 0$.

7. Assume $f \in L^{p_0}(X, \mathcal{M}, \mu)$ for some $0 < p_0 \leqslant \infty$. Prove that

$$\lim_{p \to 0} \int_X |f|^p \, d\mu = \mu(\{x \in X | f(x) \neq 0\}).$$

8. Let $X = \mathbb{N}$ and μ the counting measure on the σ-algebra $\mathcal{M} = 2^{\mathbb{N}}$. That is,

$$\mu(A) = \begin{cases} card(A), & \text{if } A \text{ is a finite set,} \\ \infty, & \text{if } A \text{ is infinite.} \end{cases}$$

(a) Show that for every function $f : \mathbb{N} \to [0, \infty]$ we have

$$\int_{\mathbb{N}} f \, d\mu = \sum_{m=1}^{\infty} f(n).$$

(b) Show that $f : \mathbb{N} \to \mathbb{C}$ is integrable with respect to the counting measure if and only if $\sum_{m=1}^{\infty} f(n)$ is absolutely convergent.

(c) Suppose that $f : \mathbb{N} \to \mathbb{R}$ is integrable with respect to the counting measure. Show that

$$\int_{\mathbb{N}} f \, d\mu = \sum_{m=1}^{\infty} f(n).$$

(d) Show that if $a_{ij} \geqslant 0$ for all $i, j = 1, 2, \ldots,$ then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$