Fall 2019

Algebra

WVU Mathematics Department

M.S. Advanced/Ph.D. Entrance Exam in Algebra

September 2019

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	☐	☐	☐	☐	☐	☐	☐	☐	☐	
Pages										
Score										

PLEASE READ THE DIRECTIONS CAREFULLY:

This exam has *three* parts:

Part A: Group Theory, **Part B**: Field and Galois Theory, **Part C**: Ring and Module Theory.

** *SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B AND C* **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded**.
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit.
  Do not submit scratchworks and solutions that are not to be graded.
  Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences using correct English grammar. *Answers, even correct, without justifications will not receive full credit*.
- **If you make use of a result which you do not prove, write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

Part A. Group Theory

Conventions.

- G denotes a group, and p denotes a prime number.
- G is a *simple* group if G is a nontrivial group whose only normal subgroups are the trivial group and the group itself, that is, if H is a normal subgroup of G, then either $H = \{e\}$ or $H = G$.
- For a subgroup H of G, the subgroup of G defined by $\mathrm{N}_G(H) = \{g \in G : gH = Hg\}$ is called the *normalizer* of H in G.

Questions.

(1) Assume G is <u>infinite</u> and $x^p = e$ for all $x \in G$.
Assume further each nontrivial and proper subgroup of G has order p, that is, if H is a subgroup of G such that $\{e\} \neq H \neq G$, then the order $|H|$ of H is p.
 (i) Prove that $p \neq 2$.
 (ii) Prove that G <u>is</u> *simple*.

(Hint: Consider the product of two subgroups of G.) □

(2) Assume G is <u>finite</u>.
Assume further G has exactly 7 Sylow 3-subgroups, each having order 27.
Prove that G is <u>not</u> *simple*.

(Hint: Consider an action of G on the set of left cosets of $\mathrm{N}_G(P)$ of a Sylow 3-subgroup P of G.)

□

(3) Assume G is <u>finite</u>.
Assume also H is a normal subgroup of G, and P is a Sylow p-subgroup <u>of H</u>.
If $N = \mathrm{N}_{\mathbf{G}}(P)$, then prove that $G = NH$. □

Part B. Field and Galois Theory

Conventions.

- $\mathbb{Q}$ denotes the set of rational numbers.
- A *Galois* extension is a field extension that is finite, normal, and separable.

Questions.

(4) Consider the polynomial $p(x) = x^3 + 9x + 6 \in \mathbb{Q}[x]$. Let $\alpha \in \mathbb{C}$ be a root of $p(x)$. Find an element $\beta \in \mathbb{Q}(\alpha)$ such that $(1+\alpha)\cdot\beta = 1$.

$\big($Hint: you may assume, without proof, that $p(x)$ is irreducible over $\mathbb{Q}$. Do not find α.$\big)$ $\square$

(5) Let p be an *odd* prime. Determine the automorphism group $\mathrm{Aut}(\mathbb{Q}(\sqrt[p]{2})/\mathbb{Q})$, that is, determine all *bijective* field homomorphisms $\mathbb{Q}(\sqrt[p]{2}) \to \mathbb{Q}(\sqrt[p]{2})$ that fix $\mathbb{Q}$. $\square$

(6) Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 4, and let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Assume the following conditions hold:

(i) $f(x)$ is *irreducible* over $\mathbb{Q}$.

(ii) If L is a subfield of K containing $\mathbb{Q}$, then $[L:\mathbb{Q}] \neq 2$.

Prove that K has degree 12 over $\mathbb{Q}$, that is, prove $[K:\mathbb{Q}] = 12$.

$\big($Hint: note that S_4 contains an isomorphic copy of the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$, and you can think of G as a subgroup of S_4. You may also use the following fact without proof: if H is a subgroup of S_4 such that $H \nsubseteq A_4$, then $[H : H \cap A_4] = 2$.$\big)$ $\square$

Part C. Ring and Module Theory

Conventions.

- R denotes a ring *(not necessarily commutative)* which has *multiplicative identity* 1 such that $1 \neq 0$.
- Moreover, all R-modules are assumed to be left modules.

Questions.

(7) Let $0 \neq a \in R$ and assume the following properties hold:

(i) If $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ is a descending chain of *left* ideals of R, then the chain stabilizes, that is, there is a positive integer s such that $I_s = I_{s+j}$ for all $j \geq 1$.

(ii) If $za = 0$ for some $z \in R$, then $z = 0$.

Prove that a is a unit in R. □

(8) Let $e \in R$ be an *idempotent* element, that is, $e^2 = e$. Set $P = Re = \{re : r \in R\}$, the cyclic *left* R-module generated by e. Prove that there exists a left R-module M such that $R = P \oplus M$. □

(9) Assume R is *commutative*. Assume further that M is a *nonzero* R-module satisfying the following:

(i) M is simple, that is, M has no R-submodules other than zero and itself.

(ii) M is faithful, that is, if $r \in R$ such that $rx = 0$ for all $x \in M$, then $r = 0$.

Prove that R is a field.

$\big($Hint: For a nonzero $m \in M$, consider the map $R \to M$ given by multiplication by m. $\big)$ □

Write **BIG** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page # of problem # | name: