

2019

## Topology

WVU Mathematics Department

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**Directions:** Solve all of the following problems. Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1. Write the solution on **one side** of the paper and stay **within the borders**. Anything written outside the borders will not be taken into account. Return your solutions with pages in correct order **arranged according to problem numbers** and together with this cover page.

- (1) Prove, or give a counterexample, to each of the following statements.
  - (a) Continuous image of a separable space is separable. (Include definition of a separable space.)
  - (b) Continuous image of a Lindelöf space is Lindelöf. (Include definition of a Lindelöf space.)
  - (c) Continuous image of a completely regular space is completely regular. (Include definition of a completely regular space.)
- (2) Let  $\langle X, d \rangle$  and  $\langle Y, \rho \rangle$  be compact metric spaces and let  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  be continuous. Show that if  $f(x) \neq g(x)$  for all  $x \in X$ , then there exists a  $\varepsilon > 0$  such that  $\rho(f(x), g(x)) > \varepsilon$ .
- (3) Let  $\{T_a\}_{a \in A}$  be a family of connected subsets of a topological space  $X$ . Let  $Y := \bigcup_{a \in A} T_a$  and assume that for every  $x, y \in Y$  there is a finite sequence  $a_0, \dots, a_n \in A$  such that  $x \in T_{a_0}$ ,  $y \in T_{a_n}$ , and  $T_{a_i} \cap T_{a_{i+1}} \neq \emptyset$  that for every  $i < n$ . Prove that  $Y$  is connected.
- (4) Let  $X$  be a regular Hausdorff space and  $A$  be an infinite subset of  $X$ . Prove that there is a sequence  $U_1, U_2, \dots$  of open subsets of  $X$  such that  $\overline{U_n} \cap \overline{U_m} = \emptyset$  when  $m \neq n$  and  $U_n \cap A \neq \emptyset$  for each  $n \in \mathbb{N}$ .  
*Hint:* Construct the sequence  $\langle U_n : n \in \mathbb{N} \rangle$  by induction so that  $A \setminus \overline{U_n}$  is infinite for every  $n$ .
- (5) Let  $X = \mathbb{N}^{\mathbb{N}}$  have the product topology assuming that  $\mathbb{N}$  has discrete topology. Let  $Y \subseteq X$  be a compact subspace of  $X$ .
  - (a) Prove that for each  $n \in \mathbb{N}$ , there is finite  $F_n \subseteq \mathbb{N}$  such that  $Y \subseteq \prod_{n \in \mathbb{N}} F_n$ .
  - (b) Show that  $X$  is not a countable union of compact subsets of  $X$ .  
*Hint:* In the proof of part (b) use part (a) and a diagonal argument to show that for every sequence  $\langle Y_n : n \in \mathbb{N} \rangle$  of compact subsets of  $X$  there is an  $x \in X$  that belongs to no  $Y_n$ .

page #

of problem #

name:

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