Spring 2017

Algebra

WVU Mathematics Department

WestVirginiaUniversity.
DEPARTMENT OF MATHEMATICS

Ph.D. Entrance Exam in Algebra

April 17, 2017

Instructions – Please read carefully before you start.

- This exam has three parts:
  Part A: Group Theory, Part B: Field and Galois Theory, and Part C: Ring and Module Theory.

- **Work on a total of 7 questions**: ***work on two questions from each part, and one additional question from any part you want (Part A or Part B or Part C), for a total of 7 questions. Clearly indicate the questions from each part you worked on.***

- **In answering the questions, state and write any theorems you use carefully and separately.**

- You should not interpret any of the exam questions as trivial by referring to a result from a textbook or any lecture notes.

- Please **write big and legibly**. Justify your arguments with complete sentences using correct grammar. Solutions, even correct, without appropriate justifications or those that cannot be read, will not receive full credit.

- No electronic devices, calculators, cell phones, etc. are allowed during the exam.

- You have **3 hours** to complete the exam (4:00 PM - 7:00 PM).

Exam Questions

PART A: Group Theory

Conventions.
- $\mathrm{Z}(G)$ denotes the *center* of a given group G.
- $|A|$ denotes the *cardinality* of a set A.
- An action of a group G on a set S is said to be *transitive* if there is only one orbit of the action.

1. Prove that there is no group G such that $|G/\mathrm{Z}(G)| = 17$.

2. Let G be a finite group with $|G| = 4 \cdot 3^s$, where $s \geq 2$ is an integer. Prove that G is *not* simple.

3. Let G be a finite group and let S be a finite set with $|S| \geq 2$. Assume G acts *transitively* on S. Prove that there is an element $g \in G$ such that $g \cdot s \neq s$ for *all* $s \in S$.

4. Let G be a finite group with $|G| = 969$. Prove that G is *solvable*. (Note $969 = 3 \cdot 17 \cdot 19$)

5. Let G be a *finite* group, and let H and K be two subgroups of G. Assume P is a Sylow p-subgroup of H, and H is a subgroup of K. If P is a *normal* subgroup of H, and H is a *normal* subgroup of K, prove that P is a *normal* subgroup of K.

PART B: Field and Galois Theory

Conventions.
- A *Galois* extension is a field extension that is finite, normal and separable.
- $\mathbb{Q}$ denotes the set of rational numbers.

1. Prove that every algebraically closed field is infinite.

2. Prove that i and $\sqrt{2}$ belong to the splitting field K of the polynomial $p(x) = x^4 + 2$ over $\mathbb{Q}$.

3. Let $E = \mathbb{Q}\left(\sqrt[3]{2}\right)$. Prove that E is *not* a subfield of $\mathbb{Q}(w)$, where w is a *primitive* nth root of unity.

4. Let $F = \mathbb{Q}$, $\alpha = \sqrt{2 + \sqrt{2}}$, and $K = F(\alpha)$. Prove that K/F is a *Galois* extension with a *cyclic* Galois group of order 4.

5. Find, *using Galois Theory*, a primitive element for $\mathbb{Q}(\sqrt{p}, \sqrt{q})$, where p and q are distinct prime integers.

PART C: Ring and Module Theory

Conventions.

- Assume all rings have multiplicative identity 1 such that $1 \neq 0$, and all modules considered are left modules.
- An element x of a ring R is said to have a *left inverse* if there exists $y \in R$ such that $yx = 1$.
- An element x of a ring R is said to be a *unit* if there exists $z \in R$ such that $xz = zx = 1$.
- A nonzero element a of a ring R is said to be a *zero-divisor* on R if there is a nonzero element $b \in R$ such that either $ab = 0$ or $ba = 0$.

1. Let R be a ring (not necessarily commutative) and let $a, b \in R$. If $1 - ba$ has a left inverse in R, prove that $1 - ab$ has a left inverse in R.

2. Let R be a ring (not necessarily commutative) and let I be a nonzero left ideal of R. Assume the following conditions hold:

   (a) if J is a nonzero left ideal of R and $J \subseteq I$, then $J = I$.

   (b) if $0 \neq x \in I$, then x is *not* a zero-divisor on R.

   Prove that every nonzero element of R is a unit.

3. Let R be a commutative ring and let I and J be nonzero ideals of R. Assume $IJ = (b)$, that is, the product IJ of I and J is a *principal* ideal of R generated by some element $b \in R$. If b is *not* a zero-divisor on R, prove that I is a finitely generated ideal of R.

4. Let R be a ring (not necessarily commutative), M be an R-module, and let N be an R-submodule of M. Assume the following condition holds for the module N: whenever there is an ascending chain of R-submodules of N of the form

   $$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq N_{n+1} \subseteq \cdots$$

   there exists a positive integer k such that $N_k = N_{k+1} = N_{k+2} = \cdots$, that is, $N_k = N_{k+i}$ for all $i \geq 1$.

   If $f : N \to M$ is a surjective R-module homomorphism, prove that f is an isomorphism.

5. Let R be a ring (not necessarily commutative). An R-module P is said to be *projective* if it is a *direct summand* of a free R-module, that is, $F = P \oplus M$ for some free R-module F and some R-module M.

   If $e \in R$ is an *idempotent* element, that is, $e^2 = e$, prove that the R-module Re is projective.

   (Recall that Re is the cyclic R-module $\{re : r \in R\}$.)