Spring 2019

Algebra

WVU Mathematics Department

M.S. Advanced/Ph.D. Entrance exam in Algebra

April 2019

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	☐	☐	☐	☐	☐	☐	☐	☐	☐	
Pages										
Score										

PLEASE READ THE DIRECTIONS CAREFULLY:

This exam has three parts:
Part A: Group Theory, **Part B**: Field and Galois Theory, **Part C**: Ring and Module Theory.

** *SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B AND C* **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded**.
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit.
Do not submit scratchworks and solutions that are not to be graded.
Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences using correct English grammar. *Answers, even correct, without justifications will not receive full credit*.
- **If you make use of a result which you do not prove, write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

Part A. Group Theory

Conventions. For a given group G,

- $|G|$ and $|x|$ denote the *order* of G, and the order of an element $x \in G$, respectively.
- $[G : H]$ denotes the *index* of a subgroup H of G in G.
- $\mathrm{Z}(G)$ denotes the *center* of G.

Questions.

(1) Let G be a *finite* group, H a *normal* subgroup of G, and let p be a prime integer. Assume $|H| = p^i$ for some $i \geq 1$. Prove that H is contained in every Sylow p-subgroup of G. □

(2) Let G be a *finite* group such that G is *not* abelian, and $|G| = p^3$ for some prime integer p. Assume H is a subgroup of G such that $|H| = p^2$.
Prove that $\mathrm{Z}(G) \subseteq H$, and H is normal in G.

(Hint: start by showing that $\mathrm{Z}(G) = \langle a \rangle$ for some $a \in G$. Suppose $a \notin H$ and seek a contradiction of the hypothesis that G is not abelian. After proving $\mathrm{Z}(G) \subseteq H$, consider the subgroup $H/\mathrm{Z}(G)$ of $G/\mathrm{Z}(G)$.) □

(3) Let G be a *finite* group, N a *normal* subgroup of G, and let P be a *Sylow* p-subgroup of G for some prime integer p. Assume $[G : N] = |P| \geq 1$. Prove that:

$$N = \{x \in G : |x| \text{ is not divisible by } p\}.$$

(Hint: start by writing $|G| = p^n \cdot m$ for some positive integer m with $\gcd(p, m) = 1$.) □

Part B. Field and Galois Theory

Conventions.

- $\mathbb{Q}$ denotes the set of rational numbers.
- A *Galois* extension is a field extension that is finite, normal, and separable.

Questions.

(4) Let E/F be a *finite* field extension, where F is an infinite field.
Assume there are finitely many intermediate fields of E/F.
Prove that E/F is a simple extension, i.e., $E = F(c)$ for some $c \in E$.

(Hint: explain first why it is enough to prove: if $a, b \in E$, then $F(a, b) = F(c)$ for some $c \in E$.)

□

(5) Let F be a field of *characteristic* p for some prime integer p.
Assume every element of F has a pth root in F, i.e., if $a \in F$, there exists $b \in F$ such that $b^p = a$.
Assume E/F is an algebraic field extension. Prove that E/F is a *separable* extension.

(Hint: let $\alpha \in E$, assume α is not separable over F, and seek a contradiction.) □

(6) Let $F = \mathbb{Q}$, $p(x) = x^4 - 14x^2 + 9 \in F[x]$, and let E be the *splitting field* of $p(x)$ over F.
Let $G = \mathrm{Gal}(E/F)$ be the *Galois group* of the field extension E/F.
Determine E explicitly, and prove that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(Hint: you may use the fact that there are only 2 groups, up to isomorphism, of order four.)

□

Part C. Ring and Module Theory

Conventions.

- R denotes a ring (not necessarily commutative) which has *multiplicative identity* 1 such that $1 \neq 0$. Moreover, all R-modules are assumed to be nonzero left modules.

Questions.

(7) Let $A = \{a \in R \mid a \neq 1\}$ and $B = \{a \in R \mid a + b = ab \text{ for some } b \in R\}$.
Prove that R is a *divison ring* if and only if $A = B$. □

(8) Assume R is commutative. Assume further that each ideal of R is a free R-module.
Prove that R is a principal ideal domain. □

(9) Assume $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of R-modules, i.e., L, M and N are R-modules, f and g are R-module homomorphisms such that f is injective, g is surjective and $\mathrm{im}(f) = \ker(g)$. Assume further that there is an R-module homomorphism $h : N \to M$ such that $gh = 1$, i.e., $g(h(n)) = n$ for all $n \in N$. Prove that $M \cong L \oplus N$, as R-modules. □

Write **big** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page # of problem # | name: