Fall 2018

Algebra

WVU Mathematics Department

West Virginia University
DEPARTMENT OF MATHEMATICS

M.S. Advanced/Ph.D. Entrance exam in Algebra

August 2018

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	☐	☐	☐	☐	☐	☐	☐	☐	☐	
Pages										
Score										

PLEASE READ THE DIRECTIONS CAREFULLY:

This exam has three parts:
Part A: Group Theory, **Part B**: Field and Galois Theory, **Part C**: Ring and Module Theory.

** *SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B AND C* **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded**.
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit.
Do not submit scratchworks and solutions that are not to be graded.
Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences using correct English grammar. *Answers, even correct, without justifications will not receive full credit*.
- **If you make use of a result which you do not prove, write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

Part A. Group Theory

Conventions. For a given group G,

- $\{e\}$ denotes the *trivial subgroup* of G.
- $|G|$ denotes the *order* of G.
- $[G : H]$ denotes the *index* of a subgroup H in G.
- $\mathsf{Z}(G)$ denotes the *center* of G.
- G' denotes the *commutator subgroup* of G (Recall: G' is generated by the set $\{xyx^{-1}y^{-1} : x, y \in G\}$).

Questions.

(1) Let G be a group such that $|G| = \mathfrak{p}^3$, where $\mathfrak{p}$ is a prime number.
Assume G is *not* abelian.
Prove that $G' = \mathsf{Z}(G)$. □

(2) Let G be a finite group of odd order, and let H be a subgroup of G such that $[G : H] = 5$.
Prove that H is a normal *subgroup of G.*
(Hint: consider an action of G on the set of left cosets of H.
Without justification, you may use the fact that S_5 has no subgroup of order 15.*)* □

(3) Let G be a finite *group such that $|G|$ is divisible by a prime number $\mathfrak{p}$.*
Assume P is a normal *subgroup of H, and H is a* normal *subgroup of K.*
Assume further P is a Sylow $\mathfrak{p}$-subgroup of H.
Prove that P is a normal *subgroup of K.* □

Part B. Field and Galois Theory

Conventions.

- $\mathbb{Q}$ denotes the set of rational numbers.
- $[F : E]$ denotes the *degree* of a given field extension F/E.
- A *Galois* extension is a field extension that is finite, normal and separable.

Questions.

(4) Let R be a commutative ring with multiplicative identity $1 \neq 0$.
Assume R is a *finite* integral domain, i.e., R is an integral domain and R is a finite set.
Prove that R must be a *field*. $\square$

(5) Let α be a *real* fourth root of 5 so that $\alpha^4 = 5$.
Consider the fields $E = \mathbb{Q}(i\alpha^2)$ and $F = \mathbb{Q}(\alpha + i\alpha)$, where i is the complex number with $i^2 = -1$.
Prove that $E \subseteq F$ and also $[F : E] = 2$, i.e., prove F/E is a field extension of degree two.
$\square$

(6) Let $F = \mathbb{Q}$, $p(x) = x^4 + 4x^2 + 2 \in F[x]$, and let K be the *splitting field* of $p(x)$ over F.
Let $G = \text{Gal}(K/F)$ be the *Galois group* of the field extension K/F.
Prove, by finding a generator of G, that G is a *cyclic* group of order 4. $\square$

Part C. Ring and Module Theory

Conventions.

- R denotes a <u>*commutative*</u> ring which has *multiplicative identity* 1 such that $1 \neq 0$.

Moreover, all modules considered over R are assumed to be nonzero left modules.

- A nonzero element $x \in R$ is called a *non zero-divisor on R* if $r = 0$ whenever $r \in R$ and $rx = 0$.
- A nonzero element $x \in R$ is called a *non zero-divisor on an R-module M* if $m = 0$ whenever $m \in M$ and $xm = 0$.
- $\text{im}(f)$ and $\ker(f)$ denote the *image* and the *kernel* of a given module homomorphism f, respectively.

Questions.

(7) Let I and J be nonzero ideals of R. Assume $IJ = (b)$, i.e., the product IJ of I and J is a *principal ideal* of R generated by a nonzero element $b \in R$. If b is a *non zero-divisor* on R, prove that I is a *finitely generated* ideal of R. □

(8) Let $x \in R$ be a nonzero element and let K, M and N be R-modules.
Assume the following hold:

(i) x is a *non zero-divisor* on M.

(ii) $0 \to K \xrightarrow{f} N \xrightarrow{g} M$ is an exact sequence, i.e., f and g are R-module homomorphisms, f is injective, and $\text{im}(f) = \ker(g)$.

Prove that the natural induced map $\overline{f} : K/xK \to N/xN$ is *injective*.

(Note: $\overline{f}(k + xK) = f(k) + xN$, i.e., $\overline{f}(\overline{k}) = \overline{f(k)}$, for each $k \in K$.) □

(9) Consider the following diagram of R-modules and R-module homomorphisms, where both *rows are exact*, and the square is *commutative*.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B & \xrightarrow{\alpha} & C \\ & & & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ 0 & \longrightarrow & L & \xrightarrow{\beta'} & M & \xrightarrow{\beta} & N \end{array}$$

In other words, we have:

(a) A, B, C, L, M, N are R-modules, and $\alpha', \alpha, \beta', \beta, g, h$ are R-module homomorphisms.
(b) $h\alpha = \beta g$ (the operation is the composition, i.e., βg means the composition of β with g).
(c) $\text{im}(\alpha') = \ker(\alpha)$ and $\text{im}(\beta') = \ker(\beta)$.
(d) α' and β' are injective.

Prove that there is an R-module homomorphism $\chi : A \to L$ making the diagram commutative, i.e., satisfying $\beta'\chi = g\alpha'$. Make sure to justify the map χ you define is <u>*well-defined*</u>. □

(Write big and legibly. Your proofs cannot be graded unless they can be read. Justify your arguments.)

page # of problem # | name: