

Fall 2018

## Algebra

WVU Mathematics Department

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M.S. Advanced/Ph.D. Entrance exam in Algebra

August 2018

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
Pages										
Score										

**PLEASE READ THE DIRECTIONS CAREFULLY:**

This exam has three parts:

**Part A:** Group Theory, **Part B:** Field and Galois Theory, **Part C:** Ring and Module Theory.

**\*\* SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B AND C \*\***

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded.***
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit. **Do not submit scratchworks and solutions that are not to be graded.** Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences using correct English grammar. *Answers, even correct, without justifications will not receive full credit.*
- **If you make use of a result which you do not prove, write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

## Part A. Group Theory

**Conventions.** For a given group  $G$ ,

- $\{e\}$  denotes the *trivial subgroup* of  $G$ .
- $|G|$  denotes the *order* of  $G$ .
- $[G : H]$  denotes the *index* of a subgroup  $H$  in  $G$ .
- $Z(G)$  denotes the *center* of  $G$ .
- $G'$  denotes the *commutator subgroup* of  $G$  (Recall:  $G'$  is generated by the set  $\{xyx^{-1}y^{-1} : x, y \in G\}$ ).

### Questions.

(1) Let  $G$  be a group such that  $|G| = p^3$ , where  $p$  is a prime number.

Assume  $G$  is *not* abelian.

Prove that  $G' = Z(G)$ . □

(2) Let  $G$  be a finite group of odd order, and let  $H$  be a subgroup of  $G$  such that  $[G : H] = 5$ .

Prove that  $H$  is a normal subgroup of  $G$ .

(Hint: consider an action of  $G$  on the set of left cosets of  $H$ .)

Without justification, you may use the fact that  $S_5$  has no subgroup of order 15.) □

(3) Let  $G$  be a finite group such that  $|G|$  is divisible by a prime number  $p$ .

Assume  $P$  is a normal subgroup of  $H$ , and  $H$  is a normal subgroup of  $K$ .

Assume further  $P$  is a Sylow  $p$ -subgroup of  $H$ .

Prove that  $P$  is a normal subgroup of  $K$ . □

## Part B. Field and Galois Theory

### Conventions.

- $\mathbb{Q}$  denotes the set of rational numbers.
- $[F : E]$  denotes the *degree* of a given field extension  $F/E$ .
- A *Galois* extension is a field extension that is finite, normal and separable.

### Questions.

(4) Let  $R$  be a commutative ring with multiplicative identity  $1 \neq 0$ .

Assume  $R$  is a *finite* integral domain, i.e.,  $R$  is an integral domain and  $R$  is a finite set.

Prove that  $R$  must be a *field*. □

(5) Let  $\alpha$  be a *real* fourth root of 5 so that  $\alpha^4 = 5$ .

Consider the fields  $E = \mathbb{Q}(i\alpha^2)$  and  $F = \mathbb{Q}(\alpha + i\alpha)$ , where  $i$  is the complex number with  $i^2 = -1$ .

Prove that  $E \subseteq F$  and also  $[F : E] = 2$ , i.e., prove  $F/E$  is a field extension of degree two. □

(6) Let  $F = \mathbb{Q}$ ,  $p(x) = x^4 + 4x^2 + 2 \in F[x]$ , and let  $K$  be the *splitting field* of  $p(x)$  over  $F$ .

Let  $G = \text{Gal}(K/F)$  be the *Galois group* of the field extension  $K/F$ .

Prove, by finding a generator of  $G$ , that  $G$  is a *cyclic* group of order 4. □

## Part C. Ring and Module Theory

### Conventions.

- $R$  denotes a commutative ring which has *multiplicative identity* 1 such that  $1 \neq 0$ . Moreover, all modules considered over  $R$  are assumed to be nonzero left modules.
- A nonzero element  $x \in R$  is called a *non zero-divisor on  $R$*  if  $rx = 0$  whenever  $r \in R$ .
- A nonzero element  $x \in R$  is called a *non zero-divisor on an  $R$ -module  $M$*  if  $xm = 0$  whenever  $m \in M$ .
- $\text{im}(f)$  and  $\text{ker}(f)$  denote the *image* and the *kernel* of a given module homomorphism  $f$ , respectively.

### Questions.

(7) Let  $I$  and  $J$  be nonzero ideals of  $R$ . Assume  $IJ = (b)$ , i.e., the product  $IJ$  of  $I$  and  $J$  is a *principal ideal* of  $R$  generated by a nonzero element  $b \in R$ . If  $b$  is a *non zero-divisor* on  $R$ , prove that  $I$  is a *finitely generated* ideal of  $R$ . □

(8) Let  $x \in R$  be a nonzero element and let  $K, M$  and  $N$  be  $R$ -modules. Assume the following hold:

(i)  $x$  is a *non zero-divisor* on  $M$ .

(ii)  $0 \rightarrow K \xrightarrow{f} N \xrightarrow{g} M$  is an exact sequence, i.e.,  $f$  and  $g$  are  $R$ -module homomorphisms,  $f$  is injective, and  $\text{im}(f) = \text{ker}(g)$ .

Prove that the natural induced map  $\bar{f} : K/xK \rightarrow N/xN$  is *injective*.

(Note:  $\bar{f}(k + xK) = f(k) + xN$ , i.e.,  $\bar{f}(\bar{k}) = \overline{f(k)}$ , for each  $k \in K$ .) □

(9) Consider the following diagram of  $R$ -modules and  $R$ -module homomorphisms, where both rows are exact, and the square is *commutative*.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B & \xrightarrow{\alpha} & C \\
 & & & & \downarrow g & & \downarrow h \\
 0 & \longrightarrow & L & \xrightarrow{\beta'} & M & \xrightarrow{\beta} & N
 \end{array}$$

In other words, we have:

- (a)  $A, B, C, L, M, N$  are  $R$ -modules, and  $\alpha', \alpha, \beta', \beta, g, h$  are  $R$ -module homomorphisms.
- (b)  $h\alpha = \beta g$  (the operation is the composition, i.e.,  $\beta g$  means the composition of  $\beta$  with  $g$ ).
- (c)  $\text{im}(\alpha') = \text{ker}(\alpha)$  and  $\text{im}(\beta') = \text{ker}(\beta)$ .
- (d)  $\alpha'$  and  $\beta'$  are injective.

Prove that there is an  $R$ -module homomorphism  $\chi : A \rightarrow L$  making the diagram commutative, i.e., satisfying  $\beta' \chi = g \alpha'$ . Make sure to justify the map  $\chi$  you define is well-defined. □

(Write big and legibly. Your proofs cannot be graded unless they can be read. Justify your arguments.)

page #

of problem #

name:

