Fall 2017

Algebra

WVU Mathematics Department

WestVirginiaUniversity.
DEPARTMENT OF MATHEMATICS

M.S. Advanced/Ph.D. Entrance exam in Algebra

September 1, 2017

Part	A			B			C			
#	1	2	3	4	5	6	7	8	9	total score
✓	☐	☐	☐	☐	☐	☐	☐	☐	☐	
pages										
score										

Directions:

Solve a total of 6 questions of the following problems, two from each part A, B and C.

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; otherwise problems 1, 2, 4, 5, 7 and 8 will be graded.

- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.

- Write the solution on **one side** of the paper and stay **within the borders**.
  Anything written outside the borders will not be taken into account.

- For each solution submitted, write in the table above how many pages you submit.
  Do not submit solutions that are not to be graded.
  Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.

PART A. Group Theory

• If G is a group and H is a subgroup of G, then $[G:H]$ denotes the index of H in G.

(1) Let G be a group and H be a subgroup of G.
   (a) If $[G : H] = 3$, then must H be *normal* in G? Give a proof, or a counterexample with justification.
   (b) If $[G:H]=2$, prove that $x^2 \in H$ for all $x \in G$. Using this, show that H is *normal* in G.

(2) Let G be a finite group of *odd order*, and H be a subgroup of G such that $[G:H]=5$.
Prove that H is normal in G.
(Hint: consider an action of G on the cosets of H. Without giving a proof, you may use the fact that S_5 has no element of order 15.)

(3) Let G be a group of order 30, P be a Sylow 3-subgroup of G and K be a Sylow 5-subgroup of G.
   (a) Prove that PK is a normal subgroup of G, and K is the unique Sylow 5-subgroup of PK.
   (b) If L is a Sylow 5-subgroup of G, prove that L is also a Sylow 5-subgroup of PK.
   (c) By using parts (a) and (b), show that K is the *unique* Sylow 5-subgroup of G.

PART B. Field and Galois Theory

• A *Galois* extension is a field extension that is finite, normal and separable. If K/F is a Galois extension, then $\mathrm{Gal}(K/F)$ denotes the Galois group of K/F.

(4) Let K/F be a field extension and let R be a ring with $F \subseteq R \subseteq K$. For each part below, give a proof, or a counterexample with justification.
   (a) Does it follow that R is a field?
   (b) Does it follow that R is a field if K is algebraic over F?

(5) Let K/F be a field extension such that $K = F(a, b)$, that is, K is generated over F by two elements $a, b \in K$. Assume a and b are algebraic over F of degrees m and n, respectively. If m and n are relatively coprime, prove that the degree $[K : F]$ of K over F is mn.

(6) Let K/F be a *Galois* extension. Prove that there exists a field L with the following properties:
   (i) $F \subseteq L \subseteq K$.
   (ii) L/F is a Galois extension and $\mathrm{Gal}(L/F)$ is abelian.
   (iii) whenever E is a field such that $F \subseteq E \subseteq K$, E/F is a Galois extension, and $\mathrm{Gal}(E/F)$ is abelian, it follows that $E \subseteq L$.

(Hint: consider the commutator subgroup of $\mathrm{Gal}(K/F)$ and its fixed field.)

PART C. Ring and Module Theory

• Assume all rings have multiplicative identity 1 such that $1 \neq 0$, and all modules considered are left modules. Moreover $\mathbb{N}$ denotes the set of positive integers.

(7) Let R be a ring, and let M and N be R-modules. Assume $f : M \to N$ and $g : N \to M$ are R-module homomorphisms such that $g \circ f$ is the identity map on M. Prove that N is the *direct sum* of $\ker(g)$ and $\operatorname{im}(f)$, where $\ker(g)$ is the kernel of g, and $\operatorname{im}(f)$ is the image of f.

(8) Let R be a ring and let $R[x]$ be the *polynomial ring* in the indeterminate x with coefficients from R. If R is a principal ideal domain, then must $R[x]$ be also a principal ideal domain?
Give a proof or a counterexample with justification.

(9) Let R be a ring and let M be a nonzero R-module. Assume the following condition holds: whenever $(M_i)_{i\in\mathbb{N}}$ is a family of R-submodules of M such that $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots$, there exists $k \in \mathbb{N}$ such that $M_k = M_{k+1} = M_{k+2} = \ldots$, that is, $M_k = M_{k+i}$ for all $i \in \mathbb{N}$. If $f : M \to M$ is an *injective* R-module homomorphism, prove that f is *surjective*.

page # of problem # | name: