

Spring 2019

## Differential Equations

WVU Mathematics Department

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### Recommended Citation

WVU Mathematics Department, "Differential Equations" (2019). *M.S. Advanced and Ph.D. Entrance Exams*. 23.

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**Differential Equations Entrance Exam, 2019S. NAME:**  
**Solve 6 problems. Indicate 6 problems of your choice.**

1. Assume that  $A$  is an  $n$  by  $n$  constant upper triangular matrix

$$A = [a_{j,k}], \quad a_{k,k} = k\sqrt{-1}, \quad k = 1, 2, \dots, n. \quad (1)$$

Consider on  $[0, \infty)$  the vector differential system

$$y' = Ay. \quad (2)$$

Show that the zero solution  $y(t) \equiv \vec{0}$  is stable but not asymptotically stable.

2. Given  $y \in \mathbb{R}^2$ ,  $\nu > 0$ ,  $\sigma < 0$  and that

$$\frac{dy(t)}{dt} = \begin{bmatrix} \sigma & \nu \\ -\nu & \sigma \end{bmatrix} y(t). \quad (3)$$

Determine a fundamental matrix solution to (3) that has real valued entries. Sketch the phase portrait of (3) as  $t \rightarrow \infty$ . Carefully explain the direction of arrows along the orbits.

3. (a) Discuss the stability of the system. Here,  $x = x(t)$ ,  $\dot{x} = \frac{dx}{dt}$  etc, and  $t \in \mathbb{R}$ .

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} x.$$

The characteristic polynomial of the above coefficient matrix  $A$  is  $\lambda^2(\lambda + 1)$ .

• In the next two questions the initial data  $x_0 \in \mathbb{R}^3$  are given at  $t = 0$ , i.e.,  $x(0) = x_0$ .

(b) Find the set of all initial data for which the solution is bounded for all  $t$ . Show that it forms a subspace of  $\mathbb{R}^3$ .

(c) Find the set of all initial data for which the solution approaches zero as  $t$  approaches infinity. Show that it forms a subspace of  $\mathbb{R}^3$ .

4. Consider a system in  $\mathbb{R}^2$ . Here,  $x = x(t)$ ,  $\dot{x} = \frac{dx}{dt}$  etc, and  $t \in \mathbb{R}$ .

$$\begin{aligned} \dot{x} &= y + x^3y, \\ \dot{y} &= x + xy. \end{aligned}$$

(a) Find the critical points and discuss their stability.

(b) Along what curves (or lines)  $\frac{dy}{dx}$  is zero or infinite? Draw these curves in the  $xy$ -plane.

(c) Sketch the phase diagram. Be sure to put "Arrows" in orbits (or paths) to indicate the direction of time.

5. Consider systems in  $\mathbb{R}^2$ . Discuss the stability of critical points. Here,  $x = x(t)$ ,  $\dot{x} = \frac{dx}{dt}$  etc, and  $t \in [0, \infty)$ .

(a)

$$\begin{aligned} \dot{x} &= 4x^2 - y^2, \\ \dot{y} &= -2x + xy - 4. \end{aligned}$$

(b)

$$\begin{aligned} \dot{x} &= 2x^2y + y^3, \\ \dot{y} &= -3xy^2 + x^3. \end{aligned}$$

6. Consider the differential equation

$$\ddot{x} = x^3 - x, \quad x = x(t) \in \mathbb{R}, \quad \ddot{x} = \frac{d^2x}{dt^2}, \quad t \in \mathbb{R}.$$

- (a) Take  $\dot{x}$  as the vertical axis and  $x$  as the horizontal axis to draw the phase diagram.
- (b) Suppose  $x(0) = -1$  and  $\dot{x}(0) = 0$ . Find the solution. Draw the orbit of the solution in the phase diagram.
- (c) Determine if there exists a solution satisfying  $x(0) = 1$  and  $x(1) = 1$ .
- (d) Determine if there exists a bounded solution satisfying  $x(0) = -1$  and  $x(1) = 1$ .

7. Given that: i)

$$A(t) = [a_{j,k}(t)], \quad a_{j,k}(t) \in C(-\infty, \infty), \quad j, k = 1, 2, \dots, n. \quad t, t_0 \in (-\infty, \infty), \quad (4)$$

ii) Exists  $\omega > 0$  such that

$$A(t + \omega) \equiv A(t), \quad \omega > 0. \quad (5)$$

Let  $M(t)$  be a square  $n$  by  $n$  fundamental matrix solution of

$$\frac{dM(t)}{dt} = A(t)M(t). \quad (6)$$

Show that:

- (a)  $M(t + \omega)$  is also a fundamental matrix solution of (6).
- (b) Show that the matrix

$$C := M^{-1}(t)M(t + \omega)$$

exists is invertible and actually is a constant independent of  $t$ .

- (c) It is known that there exists a constant matrix  $R$  such that

$$C = \exp(\omega R). \quad (7)$$

Prove that

$$P(t) := M(t)\exp(-tR)$$

is periodic with period  $\omega$  and that the transformation  $M(t) = P(t)Z(t)$  takes the differential system (6) into the system

$$\frac{dZ(t)}{dt} = RZ(t). \quad (8)$$

- (d) Determine the matrices  $C$  and  $P(t)$  with

$$\omega = 2\pi, \quad A(t) = \begin{bmatrix} \cos(t) & 1 \\ 0 & \cos(t) \end{bmatrix}, \quad M(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Can you find the matrix  $R$  as well?

8. Given

$$y' = f(t, y, p), \quad y(t_0) = \eta, \quad p \in \mathbb{R}^1, y, \eta, f \in \mathbb{R}^n. \quad (9)$$

Assume: i)  $f(t, y, p)$  is a continuous vector function in the set of points  $(t, y)$

$$REC.BOX := \{t \in I, \quad |y - \eta| \leq b\} \quad (10)$$

,  $I$  an interval

$$I = \{t \mid |t - t_0| \leq \delta\} \quad (11)$$

and

$$|f(t, y)| \leq M.$$

ii) Assume the entries of the Jacobian matrix

$$JM := \left( \frac{\partial f_i}{\partial y_j} \right) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \cdot \\ \cdot \\ \cdot \\ \nabla f_n \end{bmatrix}$$

to be continuous in the  $REC.BOX$  .

iii) Assume that there exists a solution  $y = \phi(t, t_0, \eta)$  to the initial value problem (9) on the interval

$$I = \{t \mid |t - t_0| \leq q\}, \quad q = \text{Minimum} \left\{ \delta, \frac{b}{M} \right\}.$$

Prove that the solution  $y = \phi(t, t_0, \eta)$  is unique.