Spring 2019

Differential Equations

WVU Mathematics Department

Differential Equations Entrance Exam, 2019S. NAME:
Solve 6 problems. Indicate 6 problems of your choice.

1. Assume that A is an n by n constant upper triangular matrix
$$A = [a_{j,k}], \quad a_{k,k} = k\sqrt{-1}, \; k = 1, 2, \cdots, n. \tag{1}$$
Consider on $[0, \infty)$ the vector differential system
$$y' = Ay. \tag{2}$$
Show that the zero solution $y(t) \equiv \vec{0}$ is stable but not asymptotically stable.

2. Given $y \in \mathbb{R}^2$, $\nu > 0$, $\sigma < 0$ and that
$$\frac{dy(t)}{dt} = \begin{bmatrix} \sigma & \nu \\ -\nu & \sigma \end{bmatrix} y(t). \tag{3}$$
Determine a fundamental matrix solution to (3) that has real valued entries. Sketch the phase portrait of (3) as $t \to \infty$. Carefully explain the direction of arrows along the orbits.

3. (a) Discuss the stability of the system. Here, $x = x(t)$, $\dot{x} = \frac{dx}{dt}$ etc, and $t \in R$.
$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} x.$$
The characteristic polynomial of the above coefficient matrix A is $\lambda^2(\lambda + 1)$.

   • In the next two questions the initial data $x_0 \in R^3$ are given at $t = 0$, i.e., $x(0) = x_0$.

   (b) Find the set of all initial data for which the solution is bounded for all t. Show that it forms a subspace of R^3.

   (c) Find the set of all initial data for which the solution approaches zero as t approaches infinity. Show that it forms a subspace of R^3.

4. Consider a system in R^2. Here, $x = x(t)$, $\dot{x} = \frac{dx}{dt}$ etc, and $t \in R$.
$$\begin{aligned} \dot{x} &= y + x^3 y, \\ \dot{y} &= x + xy. \end{aligned}$$
(a) Find the critical points and discuss their stability.

   (b) Along what curves (or lines) $\frac{dy}{dx}$ is zero or infinite? Draw these curves in the xy-plane.

   (c) Sketch the phase diagram. Be sure to put "Arrows" in orbits (or paths) to indicate the direction of time.

5. Consider systems in R^2. Discuss the stability of critical points. Here, $x = x(t)$, $\dot{x} = \frac{dx}{dt}$ etc, and $t \in [0, \infty)$.

   (a)
$$\begin{aligned} \dot{x} &= 4x^2 - y^2, \\ \dot{y} &= -2x + xy - 4. \end{aligned}$$
(b)
$$\begin{aligned} \dot{x} &= 2x^2 y + y^3, \\ \dot{y} &= -3xy^2 + x^3. \end{aligned}$$

6. Consider the differential equation

$$\ddot{x} = x^3 - x, \quad x = x(t) \in R, \ \ddot{x} = \frac{d^2x}{dt^2}, \ t \in R.$$

(a) Take $\dot{x}$ as the vertical axis and x as the horizontal axis to draw the phase diagram.

(b) Suppose $x(0) = -1$ and $\dot{x}(0) = 0$. Find the solution. Draw the orbit of the solution in the phase diagram.

(c) Determine if there exists a solution satisfying $x(0) = 1$ and $x(1) = 1$.

(d) Determine if there exists a bounded solution satisfying $x(0) = -1$ and $x(1) = 1$.

7. Given that: i)

$$A(t) = [a_{j,k}(t)], \quad a_{j,k}(t) \in C(-\infty, \infty), \ j, k = 1, 2, \cdots, n. \ t, t_0 \in (-\infty, \infty), \tag{4}$$

ii) Exists $\omega > 0$ such that

$$A(t + \omega) \equiv A(t), \ \omega > 0. \tag{5}$$

Let $M(t)$ be a square n by n fundamental matrix solution of

$$\frac{dM(t)}{dt} = A(t)M(t). \tag{6}$$

Show that:

(a) $M(t + \omega)$ is also a fundamental matrix solution of (6).

(b) Show that the matrix

$$C := M^{-1}(t)M(t + \omega)$$

exists is invertible and actually is a constant independent of t.

(c) It is known that there exists a constant matrix R such that

$$C = exp(\omega R). \tag{7}$$

Prove that

$$P(t) := M(t)exp(-tR)$$

is periodic with period ω and that the transformation $M(t) = P(t)Z(t)$ takes the differential system (6) into the system

$$\frac{dZ(t)}{dt} = RZ(t). \tag{8}$$

(d) Determine the matrices C and $P(t)$ with

$$\omega = 2\pi, \ A(t) = \begin{bmatrix} cos(t) & 1 \\ 0 & cos(t) \end{bmatrix}, \ M(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Can you find the matrix R as well?

8. Given

$$y' = f(t, y, p), \quad y(t_0) = \eta, \quad p \in \mathbb{R}^1, y, \eta, f \in \mathbb{R}^n. \tag{9}$$

Assume: i) $f(t, y, p)$ is a continuous vector function in the set of points (t, y)

$$REC.BOX := \{t \in I, \quad |y - \eta| \leq b\} \tag{10}$$

, I an interval

$$I = \{t|\,|t - t_0| \leq \delta\} \tag{11}$$

and

$$|f(t, y)| \leq M.$$

ii) Assume the entries of the Jacobian matrix

$$JM := (\frac{\partial f_i}{\partial y_j}) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \cdot \\ \cdot \\ \cdot \\ \nabla f_n \end{bmatrix}$$

to be continuous in the $REC.BOX$.

iii) Assume that there exists a solution $y = \phi(t, t_0, \eta)$ to the initial value problem (9) on the interval

$$I = \{t|\,|t - t_0| \leq q\}, \quad q = Minimum\,\{\delta, \frac{b}{M}\}.$$

Prove that the solution $y = \phi(t, t_0, \eta)$ is unique.