Spring 2018

Differential Equations

WVU Mathematics Department

Odes-phd entrance exam spring 2018

Name (Print).___________________

Show all Work. Draw and Explain. All problems carry the same weight.
Do 6 out of the 8 problems and mark the 6 problems you want to be checked.

1. (a) Find the general solution and discuss the stability of the system

$$\dot{x} = Ax = \begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} x.$$

(b) Discuss the stability of the system for $t \geq 1$.

$$\dot{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} -x_1 + t^{-2}x_1 + x_2 - 2x_3 - 2e^{-t}x_3 + e^{t^2} \\ t^{-2}x_2 - 2x_3 + t \\ e^{-2t}x_1 - x_3 + 1 \end{bmatrix}.$$

2. Consider

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x^3 = 0.$$

(a) Show whether the zero solution is stable, asymptotically stable, or unstable.
(b) Show whether the equation has a limit cycle.

3. Discuss the stability of critical points.

(a)

$$\begin{aligned} \dot{x} &= -y + xy^2, \\ \dot{y} &= x - 2x^2y. \end{aligned}$$

(b)

$$\begin{aligned} \dot{x} &= y^3 + x^2y, \\ \dot{y} &= x^3 - 2xy^2. \end{aligned}$$

4. (a) Discuss the stability of critical points.

$$\begin{aligned} \dot{x} &= y(1 + x^3), \\ \dot{y} &= x(y + 1). \end{aligned} \qquad (1)$$

(b) Sketch the phase diagram of the system (1).

5. (a) Determine the vector solution to the initial value problem

$$y' = \frac{dy}{dt} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} y - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; y(0) = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \eta \end{bmatrix} \qquad (2)$$

as a finite sum of vectors.

(b) Determine all values of the parameters $\alpha, \beta, \gamma, \eta \in \mathbb{R}$ that render the solution in a) bounded on $[0, \infty)$. Are there any parameters $\alpha, \beta, \gamma, \eta \in \mathbb{R}$ that render a solution to 2 periodic on $(-\infty, \infty)$? If yes determine all of them.

6. (a) Formulate and prove the Gronwal inequality. Explain the importance of the Gronwal inequality. What is it good for?

   b) Under what conditions on $y(t)$ does the inequality

$$y(t) \leq \int_0^t [sin^2 y(u)]du$$

   imply that $y(t) \equiv 0$ for $t \in [0,1]$? Justify your answer.

7. Given a differential system

$$y' = f(t, y).$$

   (a) Formulate an existence and uniqueness theorem for the above nonlinear system. Explain the method of successive approximations.

   (b) Prove that all solutions to

$$y_1' = cos(ty_1y_2),\ y_2' = sin(y_1^2 + y_2^3 + t) \tag{3}$$

   exist on $(-\infty, \infty)$.

8. Given the scalar initial value problem

$$2x'' + 6x^5 = 0,\ x(0) = \alpha,\ x'(0) = \beta,\ \alpha, \beta \in \mathbb{R}. \tag{4}$$

   (a) Show that if $x(t)$ is a solution then

$$[x'(t)]^2 + x^6(t) = \beta^2 + \alpha^6. \tag{5}$$

   (b) Show that (4) possesses a bounded solution with a bounded derivative on $(-\infty, \infty)$.

   (c) It is given further that $\alpha > 0, \beta > 0$. Show that the solution $x(t)$ attains a (relative) local maximum at a time $t_{max} > 0$. Determine precisely the values of $x(t_{max})$, $x'(t_{max})$, $x''(t_{max})$ in terms of α, β. Show that there exists a time $t_{min} > t_{max} > 0$ where $x(t)$ attains a local minimum. Determine precisely the values of $x(t_{min})$, $x'(t_{min})$, $x''(t_{min})$.

   (d) Sketch the orbit of this solution for $t_{max} \leq t \leq t_{min}$ in the phase space. Also sketch in a (t, x) plane the graph of $x(t)$ on the interval $t_{max} \leq t \leq t_{min}$. What is the sign of $x'(t)$ on this interval? Conclude that there is a one to one correspondence between the values of t and $x(t)$ on $t_{max} \leq t \leq t_{min}$.