Spring 2013

Differential Equations

WVU Mathematics Department

Graduate exam 2013, ODE's

Do any 6 problems. All problems carry the same weight. Explain your solutions.

1. Solve the initial value problem

$$\dot{y} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} y - \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad y(7) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Determine $\lim_{t\to\infty} y(t)$.

2. Let A be a constant n by n matrix with eigenvalues λ_j, $j = 1, 2, \ldots, n$. Let $\rho := \max\{Re(\lambda_j) : j = 1, 2, \ldots, n\}$. Consider a fixed vector solution of

$$\dot{y} = Ay(t)$$

on the interval $[a, \infty)$. Prove that for every $\epsilon > 0$, there exists $K > 0$ such that

$$|y(t)| \le Ke^{(\rho+\epsilon)t} \quad \text{for all} \quad t \in [a, \infty),$$

where $|\cdot|$ is a suitable norm.

3. Consider the autonomous differential system

$$\dot{y} = f(y), \quad \text{where} \quad f \in C^1(\mathbb{R}^n). \tag{1}$$

Let $y(t)$ be a continuous $n \times 1$ column vector solution of (1) on (a, b). Prove the following:

(i) Assume $\dot{y}$ is not identically zero and that $\lim_{t\to b^+} y(t) = L$, where L is a critical point of (1). Then $b = \infty$.

(ii) $y(t + \omega)$ is also a vector solution of (1) on $(a + \omega, b + \omega)$.

4. (i) Calculate the Jacobian matrix of f if

$$f(t, y) = A(t)y + g(t).$$

(*ii*) Prove in detail (using the theorem of existence and uniqueness for general systems), that if $A(t)$ and $g(t)$ possess continuous entries on the closed interval $[c,d]$, then every initial value problem

$$\dot{y} = A(t)y + g(t), \quad y(t_0) = \eta, \quad t_0 \in [c,d],\ \eta \in \mathbb{R}^n$$

possesses a unique solution on $[c,d]$.

5. Discuss the stability of the zero solution for

$$\dot{x} = -x\cos y + xy + ye^{-t}, \quad \dot{y} = \sin x - 2y + 2x\sin y + x\sin\frac{1}{1+t^2}.$$

6. (*i*) (2 points) Show that the origin is the only equilibrium solution for

$$\dot{x} = -x^3 - xy^4, \quad \dot{y} = -y^3 - x^2y.$$

(*ii*) Show that the equilibrium is *globally* asymptotically stable (which means that every solution, no matter where it starts, approaches this equilibrium forward in time).

(*iii*) Does this system have nonconstant *periodic solutions*?

7. Consider the scalar equation $\ddot{x} + 2x^3 = 0$.

(*i*) Sketch the phase diagram, find the critical points and study their stability.

(*ii*) Assume $x(0) = 1$, $\dot{x}(0) = \sqrt{3}$. Prove that the time it takes the corresponding orbit to cross the x-axis in phase plane is $t_0 = \displaystyle\int_0^{\sqrt{2}} \frac{ds}{\sqrt{4-s^4}}$.

(*iii*) Show that t_0 defined at (c) is finite (note that the integral above is improper, so it is necessary to prove that it is convergent).

8. Consider the system

$$\dot{x} = 2x + y - 2x^3 - 3xy^2, \quad \dot{y} = -2x + 4y - 2x^2y - 4y^3.$$

(*i*) Find all critical points and discuss their stability. *Hint: First use the Lyapunov function* $V(x,y) = 2x^2 + y^2$ *to prove that if there are other critical points beside the origin, then they must lie on the unit circle.*

(*ii*) Are there any periodic solutions? If so, approximately where are their orbits located?