Spring 2012

Differential Equations

WVU Mathematics Department

GRADUATE TEST-SPRING 2012, ODE'S

Solve any 6 (six) problems

1. Let I_n be the $n \times n$ identity matrix and J be the $n \times n$ Jordan block

$$J = \lambda I_n + H$$

with λ a given real constant and H the $n \times n$ matrix $H := (h_{j,k}) = \begin{bmatrix} 0 & 1 & 0 & & & 0 & 0 \\ & 0 & 1 & & & & 0 \\ & & & \cdot & \cdot & & \\ \cdot & & & & \cdot & \cdot & \\ & & & & & \cdot & \cdot \\ 0 & 0 & 0 & & & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}.$

Namely,

$H := (h_{j,k}), j, k = 1, \ldots, n, h_{j,k} = 1 \; if \; k = j+1, j = 1, 2, \ldots, n-1, \; h_{j,k} = 0 \; if \; k \neq j+1.$

a) Prove that $M = \exp(tJ)$ is a fundamental matrix solution of the system

$$\frac{d}{dt} M(t) = J M(t).$$

b) Prove that

$$\exp(tJ) = \exp(t\lambda) \Big(I_n + \sum_{k=1}^{n-1} \frac{t^k}{k!} H^k \Big).$$

c) Determine the elements of the $n \times n$ matrix

$$\lim_{t \to \infty} t^{-(n-1)} \Big(I_n + \sum_{k=1}^{n-1} \frac{t^k}{k!} H^k \Big).$$

2. A differential system is called complete if for any $y_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}$ there exists a solution $y(t)$ defined for all $t \in \mathbb{R}$ and such that $y(t_0) = y_0$. Given the differential system with unknown $y = (y_1, y_2) \in \mathbb{R}^2$

$$\dot{y}_1 = [\cos(t + y_1 y_2)]^{23} + 3\sin(y_1^5 + y_2^{2012}),$$

$$\dot{y}_2 = \sin[\log(1 + y_1^2 y_2^4)] + 3\cos(y_1^5 + y_2^{2012}),$$

prove that it is complete.

3. Given the autonomous differential system

$$\dot{y} = f(y), \;\; \text{where } f \in C^1(\mathbb{R}^n),$$

let $y(t)$ be a $n \times 1$ column real-valued vector solution defined on $\mathbb{R}$. Prove that if $y(t_0 + P) = y(t_0)$ for some $t_0 \in \mathbb{R}$ and some $P > 0$, then

$$y(t + P) = y(t), \;\; \text{for all } t \in \mathbb{R},$$

i.e. $y(t)$ is a periodic function of $t \in \mathbb{R}$.

4. Given the autonomous differential system

$$\dot{y} = f(y), \text{ where } f \in C^1(\mathbb{R}^n), \tag{0.1}$$

let $y(t)$ be an $n \times 1$ column real-valued vector solution defined on (a, b). Prove the following:

(i) If $\lim_{t\to\infty} y(t) = L$ (L being a constant vector in $\mathbb{R}^n$), then L is a critical point for the system.

(ii) Assume $\dot{y}$ is not identically zero and that $\lim_{t\to b^+} y(t) = L$, where L is a critical point for the system. Then $b = \infty$.

5. Let $f(x, y) = x^2y + y^3$ and $g(x, y) = y(1 - x^2 - 2y^2) - 2x^3 - 2xy^2$.
(a) Prove that the system below has periodic solutions

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y).$$

(b) Prove that the system below has non-periodic, non-constant solutions

$$\dot{x} = -x + f(x, y), \quad \dot{y} = g(x, y).$$

6. Consider the system

$$\dot{x} = x + xy, \quad \dot{y} = y + xy.$$

(a) Find the equilibria and study their stability.
(b) Prove that $x = 0$, $y = 0$ and $x = y$ are the only invariant lines.
(c) Find the null and vertical (possibly curvilinear) clines.
(d) Draw the phase diagram and use arrows to indicate the direction of increasing time.

7. (a) Solve the initial value problem

$$\dot{x} = x + y + z, \quad \dot{y} = y + z, \quad \dot{z} = -z \text{ with } x(0) = y(0) = z(0) = 1.$$

(b) Find all initial vectors $(x(0), y(0), z(0))$ for which the solution converges to zero as t converges to infinity.
(c) Find all initial vectors $(x(0), y(0), z(0))$ for which the solution converges to zero as t converges to negative infinity.

8. Consider the equation

$$\ddot{x} + \cos x = 0.$$

(a) Find all the critical points and discuss their stability.
(b) Find the level set of the Hamiltonian which contains all the unstable critical points, i.e. find the constant $c \in \mathbb{R}$ such that the planar curve of equation $H(x, y) = c$ contains all the unstable critical points.
(c) Draw the phase diagram and use arrows to indicate the direction of increasing time (independent variable).