Fall 2013

Differential Equations

WVU Mathematics Department

ENTRANCE EXAM: Differential Equations, Fall 2013.
Solve any 6 (six) problems. All problems carry the same weight.

1. Given the initial value problem

$$y' = f(t, y, p), \quad y(t_0) = \eta,$$
$$t, t_0, p \in \mathbb{R}^1, \qquad y, \eta, f \in \mathbb{R}^n.$$

Assume that $f(t, y, p)$ is a continuous vector function and that the entries of the Jacobian matrix

$$Jf := \left(\frac{\partial f_i}{\partial y_j}\right)_{ij} = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \cdot \\ \cdot \\ \cdot \\ \nabla f_n \end{bmatrix}$$

are continuous in the set of points (t, y, p), where

$$t, t_0, p \in \mathbb{R}^1, \qquad y, \eta, f \in \mathbb{R}^n.$$

Consider the solution $y = \phi(t, t_0, \eta, p)$ as a vector function of the variables (t, t_0, η, p). Determine an equation that is satisfied by a) $\frac{\partial \phi}{\partial t_0}$, b) $\frac{\partial \phi}{\partial p}$. What information could be derived from these equations? Why is that information important?

2. Convert the scalar ode $x'' + x^7 = 0$ into a vector system of differential equations. Show, that a solution to any initial value problem of the ode exists on any interval $[a, b]$.

3. a) Explain the essence of the method of successive approximations and under what assumptions it is valid. b) Calculate the first 3 successive approximations for the initial value problem

$$y' = \begin{bmatrix} -y_1 + y_2^2 \\ y_1 + y_2 \end{bmatrix}, \; y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

4. Provide the best estimates for the maximal intervals of existence of solutions of

$$y' = \begin{bmatrix} (t+5)^{-2} & t \\ t^2 & (t-1)^{-3} \end{bmatrix} y, \quad y(t_0) = \eta,$$

with a) $t_0 = -10$, b) $t_0 = 0$, c)$t_0 = 2$.

5. (*a*) Find and classify all the equilibria of the system

$$\dot{x} = -3y + xy - 4, \quad \dot{y} = x^2 - xy.$$

(*b*) Sketch the phase diagram.

6. Consider the system

$$\dot{x} = x - xy, \quad \dot{y} = y + xy.$$

(*a*) Find the equilibria and study their stability.
(*b*) Find all the invariant lines.
(*c*) Find the null and vertical (possibly curvilinear) clines.
(*d*) Draw the phase diagram and use arrows to indicate the direction of increasing time.

7. Consider the system

$$\dot{x} = 2x + y - 2x^3 - 3xy^2, \quad \dot{y} = -2x + 4y - 2x^2y - 4y^3.$$

(*a*) (6 points) Find all critical points and discuss their stability. *Hint: First use the Lyapunov function* $V(x, y) = 2x^2 + y^2$ *to prove that if there are other critical points beside the origin, then they must lie on the unit circle.*
(*b*) (4 points) Are there any periodic solutions? If so, approximately where are their orbits located?

8. Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where A is an $n \times n$ nonsingular, antisymmetric (i.e. $A^T = -A$), real matrix.
(*a*) Show that every solution satisfies $|\mathbf{x}(t)| = const$ for all $t \in \mathbb{R}$.
(*b*) Find all critical points and study their stability.
(*c*) If $n = 2$, are there any non-periodic solutions?