Fall 2012

Differential Equations

WVU Mathematics Department

ENTRANCE EXAM: *Differential Equations* (August 30, 2012)

There are 8 (eight) problems on this test. Solve any 6 (six) of them.
Clearly motivate your answers/solutions.

1. Let $M(t)$ be a square matrix and consider the matrix differential equation

$$\frac{dM(t)}{dt} = AM(t), \ A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Determine a fundamental solution.

(b) Determine all column vector solutions $y(t)$ to

$$\frac{dy(t)}{dt} = Ay(t)$$

that converge to the origin as $t \to \infty$.

(c) Determine all solutions that are bounded on $(-\infty, \infty)$.

2. A differential system is called complete if all of its real-valued solutions to all of its initial value problems exist for all $t \in \mathbb{R}$. Consider the differential system $(y_1, y_2 \in \mathbb{R})$

$$\begin{cases} \dot{y}_1 = 3\cos\big(\exp(y_1 y_2)\big) + y_1 y_2 \big(1 + y_1^2 y_2^2\big)^{-1}, \\ \dot{y}_2 = \cos(1 + y_1^2 y_2^4) + \sin(y_1^5 + y_2^{20}). \end{cases} \tag{1}$$

Prove that (1) is complete.

3. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a continuously differentiable vector field $f = f(y,p)$ with $y \in \mathbb{R}^n, \ p \in \mathbb{R}$. Let $y(t,p)$ be an n by 1 vector valued solution of the IVP:

$$\frac{dy}{dt}(t,p) = f(y(t,p),p), \quad t, \, p \in \mathbb{R}, \text{ with } y(t_0,p) = y_0 \in \mathbb{R}^n \text{ for all } p \in \mathbb{R}. \tag{2}$$

Prove that

$$\frac{\partial y}{\partial p}(t,p) = \int_{t_0}^{t} \frac{\partial f}{\partial p}(y(s,p),p)ds + \int_{t_0}^{t} J\frac{\partial y}{\partial p}(s,p)ds,$$

where J is the Jacobian matrix

$$J = (a_{k,j}), \ a_{k,j} := \frac{\partial f_k}{\partial y_j}(y,p), \ k, j = 1, \cdots, n.$$

Hint: Convert (2) into an integral equation.

4. Let

$$A(t) := \begin{bmatrix} t & \sin t \\ \cos t & -t \end{bmatrix}.$$

Denote by $M(t)$ a 2 by 2 matrix solution of the linear matrix equation

$$\frac{dM(t)}{dt} = A(t)M(t), \quad -\infty < t < \infty.$$

Prove that $det\, M(t)$, the determinant of $M(t)$, is a constant on $(-\infty, \infty)$.

5. (a) Find and classify all the equilibria of the system

$$\dot{x} = -6y + 2xy - 8, \quad \dot{y} = -x^2 + y^2.$$

(b) Sketch the phase diagram of the system from (a) (make sure to draw arrows to show direction of increasing time).

6. Consider the system

$$\dot{x} = x - xy, \quad \dot{y} = y - xy.$$

(a) Find the equilibria and study their stability.

(b) Prove that $x = 0$, $y = 0$ and $x = y$ are the only invariant lines.

(c) Find the null and vertical (possibly curvilinear) clines.

(d) Draw the phase diagram and use arrows to indicate the direction of increasing time.

7. Consider the system

$$\dot{x} = x - y - 2x^2, \quad \dot{y} = x + y - 2y^2.$$

(a) Discuss the stability of the origin.

(b) Determine if there is a periodic solution. In either case, show your proof.

8. Consider the initial value problem (IVP)

$$\ddot{x} - 2x - 2x^3 = 0, \quad x(0) = \alpha, \ \dot{x}(0) = \beta.$$

(a) Use the Hamiltonian energy of the solution $x(t)$ to show that if $\alpha = 1, \ \beta = 0$, then $x(t) \geq 1$ on its maximal interval of existence.

(b) Find the solution with $\alpha = 0, \ \beta = 1$. What is its maximal interval of existence? (Recall: the *maximal interval of existence* is the interval around $t = 0$ beyond which the solution is undefined.)