Fall 2018

Differential Equations

WVU Mathematics Department

Differential Equations Entrance Exam, 2018f. NAME:
Solve 6 problems. Indicate 6 problems of your choice.

1. (a) Assume that the scalar function $w(t)$ and it derivative $w'(t)$ are continuous on $[a, \infty)$ and that
$$lim_{t\to\infty} w(t) = L_1,\ lim_{t\to\infty} w'(t) = L_2. \tag{1}$$
Prove that $L_2 = 0$. Hint: you may use a relation like $w(t) = w(a) + \int_a^t w'(s)ds$ and argue by contradiction.

   (b) Given the autonomous differential system
$$\frac{dy(t)}{dt} = f(y),\ f(y) \in C^1(\mathbb{R}^n). \tag{2}$$
Let $y(t)$ be a continuous n by 1 column vector solution of (2) on (a, b). Prove the following. If $lim_{t\to\infty} y(t) = L$, L being a constant vector, then L is a critical point of (2). Hint:use i)

   (c) Assume $\frac{dy(t)}{dt}$ is not identically zero and that $lim_{t\to b^+} y(t) = L$, where L is a critical point of (2). Then $b = \infty$.

2. (a) Given the differential system
$$\frac{dy(t)}{dt} = f(t, y). \tag{3}$$
Formulate an existence theorem for solutions of initial value problems to (3). Discuss and explain the method of successive approximations. What are its goals? What assumptions on (3) are needed to make the method of successive approximations guarantee a unique solution. How is the Gronwall lemma used in this method?

   (b) Estimate an interval of existence for the initial value problem
$$y_1' = y_1y_2,\ y_2' = \frac{y_2}{10 - y_1},\ y_1(0) = 1,\ y_2(0) = 1. \tag{4}$$

   (c) Does
$$y_1' = y_1y_2,\ y_2' = \frac{y_2}{10 - y_1}$$
(4) possess solutions on $(-\infty, \infty)$? If yes which are they?

3. Given the differential system
$$y' = Jy, J = \begin{bmatrix} 2i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}. \tag{5}$$
(I) Determine a fundamental set of vector solutions on $(-\infty, \infty)$.

   (II) Determine all vector solutions of (5):

      (a) That are bounded for $t > 0$.

      (b) That are bounded on $t < 0$.

      (c) That are periodic non constant solutions and determine the smallest positive period.

4. (i) Formulate and prove the Gronwall inequality.

   (ii) Show that if $f(t)$ is a non negative continuous function on $[0,\ 1]$ and if

$$f(t) \leq \int_0^t 3[|sin(f(s))|]ds,\ t \in [0,\ 1],$$

   then $f(t) \equiv 0$ on $[0,\ 1]$.

5. Consider

$$x'' + (x^2 + (x')^2 - 1)x' + x^3 = 0,$$

   where $x' = \frac{dx}{dt}$ and so forth.

   (a) Show whether the zero solution is stable, asymptotically stable, or unstable.

   (b) Show whether the equation has a limit cycle.

6. (a) Discuss the stability of critical points.

$$\begin{aligned} x' &= y + xy, \\ y' &= xy^3 + x. \end{aligned} \tag{6}$$

   (b) Sketch the phase diagram of the system (6). Indicate the increasing direction of time.

   (c) Among the following lines which are invariant? If none are invariant, state so.

   (i) $y = 0$, (ii) $x = -1$, (iii) $x = 0$, (iv) $y = -1$.

7. (a) Verify that $y = t$ $(t > 0)$ is a solution for the second order differential equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0.$$

   (b) Find the general solution for

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3,\ t > 0.$$

8. Consider

$$x' = f(x), \tag{7}$$

   where $x \in R^n$, f is continuously differentiable, $f(0) = 0$, and $x = 0$ is an isolated critical point. Suppose that the linearized system is given by

$$x' = Ax, \tag{8}$$

   where

$$A = \frac{\partial f}{\partial x}(0)$$

   is the Jacobian of f evaluated at 0. Suppose the zero solution of (8) is stable. Either prove that the zero solution of (7) is stable or show a counter example.