

Fall 2017

## Differential Equations

WVU Mathematics Department

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PhD Entrance Exam, Differential Equations; Fall 2017

*INSTRUCTIONS (Please read carefully!): Solve any 6 (SIX) PROBLEMS, and clearly indicate which ones are to be graded by encircling the problem numbers on these exam pages! ONLY THE FIRST 6 (SIX) ENCIRCLED PROBLEMS WILL BE GRADED!*

Below  $\dot{f}$  represents the derivative of the function  $f$  with respect to the real variable  $t$ .  $C(\mathbb{R}^d)$  denotes the set of continuous functions on  $\mathbb{R}^d$ , while  $C^k(\mathbb{R}^d)$  denotes the set of functions on  $\mathbb{R}^d$  with continuous derivatives of up to and including  $k^{\text{th}}$  order.

1. (a) Solve the initial value problem

$$\dot{y} = Ay - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

expressing your solution in finite terms with  $A = TJT^{-1}$ , where

$$T = 2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Determine  $\lim_{t \rightarrow \infty} y(t)$ .

2. Given the autonomous differential system

$$\dot{y} = f(y), \quad f \in C^1(\mathbb{R}^n), \quad (1)$$

let  $y(t)$  be a continuous  $n \times 1$  column vector solution of (1) on the interval  $(a, b)$ . Prove the following:

(i) If  $b = \infty$  and  $\lim_{t \rightarrow \infty} y(t) = L$  for some constant vector  $L$ , then  $L$  is a critical point of (1).

(ii) Assume the derivative  $\dot{y}$  is not identically zero on  $(a, b)$  and that  $\lim_{t \rightarrow b^+} y(t) = L$ , where  $L$  is a critical point of (1). Prove that  $b = \infty$ .

(iii)  $y(t + \omega)$  is also a vector solution of (1) on  $(a + \omega, b + \omega)$ .

3. (a) Estimate an interval of existence for the initial value problem

$$\dot{y}_1 = \frac{y_2}{1 - y_1}, \quad \dot{y}_2 = \frac{y_1}{1 + y_2}, \quad y_1(0) = 2, \quad y_2(0) = 1. \quad (2)$$

(b) Set up an equivalent integral equation for (2) and determine the first two successive approximations for the solution.

4. Provide examples for the following:

(a) An example of an initial value problem  $\dot{y} = f(y)$ ,  $y(0) = 0$  with  $f \in C(\mathbb{R})$  such that the initial value problem does not have a unique solution.

(b) An example of an initial value problem  $\dot{y} = f(y)$ ,  $y(0) = y_0$  with  $f \in C(\mathbb{R})$ , with solutions that do not exist on the whole  $\mathbb{R}$

(c) An example of a scalar linear homogeneous singular differential equation such that all solutions exist on the whole  $\mathbb{R}$ .

5. Discuss the stability of the zero solution for

$$\dot{x} = x(\cos y - y) - y(1 + t^2)^{-1}, \quad \dot{y} = -2 \sin x + y - 4x \sin y + 2x(1 + t)^{-2}.$$

6. Consider the differential system

$$\dot{\mathbf{x}} = -A(t)\mathbf{x} - |\mathbf{x}|\mathbf{x}, \quad t \geq 0, \quad (3)$$

where  $A : [0, \infty) \rightarrow \mathbb{R}^{n \times n}$  is a matrix-valued continuous function, and  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{R}^n$ .

(a) Assume that there exists a constant  $0 < B < \infty$  such that all the entries of the matrix  $A(t)$  satisfy

$$|A_{ij}(t)| \leq B \text{ for all } t \in [0, \infty) \text{ and all } 1 \leq i, j \leq n.$$

Prove that for any  $\mathbf{x}_0 \in \mathbb{R}^n$  the initial value problem given by (3) and  $\mathbf{x}(0) = \mathbf{x}_0$  has a unique solution defined on  $[0, \infty)$ .

(b) Assume further that there exists a continuous function  $\lambda : [0, \infty) \rightarrow (0, \infty)$  such that

$$\int_0^\infty \lambda(t) dt = \infty$$

and

$$\mathbf{v}^T A(t)\mathbf{v} \geq \lambda(t)|\mathbf{v}|^2 \text{ for all } t \in [0, \infty) \text{ and all column vectors } \mathbf{v} \in \mathbb{R}^n,$$

where  $\mathbf{v}^T$  and  $|\mathbf{v}|$  denote the transpose and the Euclidean norm of  $\mathbf{v}$ , respectively. Prove that the zero solution of (3) is globally asymptotically stable.

7. Consider a continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that the scalar differential equation

$$\dot{x} = f(t, x) \tag{4}$$

has the following property: for all  $t_0, x_0 \in \mathbb{R}$ , the initial value problem (4) together with  $x(t_0) = x_0$  has at least one solution defined on the whole  $\mathbb{R}$ .

Prove the following statement: for any  $c_1, c_2 \in \mathbb{R}$  and any solutions  $x_1, x_2$  of (4) the function  $c_1x_1 + c_2x_2$  is also a solution of (4) if and only if there exists a continuous function  $a : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(t, x) = a(t)x$  for all  $t, x \in \mathbb{R}$ .

8. Consider the system of scalar differential equations

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y),$$

where  $f, g$  are continuous real-valued functions defined on  $\mathbb{R}^2$  for which there exists a *nonnegative* real-valued function  $h$  such that

$$x[2f(x, y) + g(x, y)] + y[f(x, y) + 2g(x, y)] = h(x, y)(1 - x^2 - y^2) \text{ for all } x, y \in \mathbb{R}.$$

Prove that the system has periodic solutions.