Fall 2017

Differential Equations

WVU Mathematics Department

PhD Entrance Exam, Differential Equations; Fall 2017

INSTRUCTIONS (Please read carefully!): Solve any 6 (SIX) PROBLEMS, and clearly indicate which ones are to be graded by encircling the problem numbers on these exam pages! ONLY THE FIRST 6 (SIX) ENCIRCLED PROBLEMS WILL BE GRADED!

Below $\dot{f}$ represents the derivative of the function f with respect to the real variable t. $C(\mathbb{R}^d)$ denotes the set of continuous functions on $\mathbb{R}^d$, while $C^k(\mathbb{R}^d)$ denotes the set of functions on $\mathbb{R}^d$ with continuous derivatives of up to and including k^{th} order.

1. (a) Solve the initial value problem

$$\dot{y} = Ay - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

expressing your solution in finite terms with $A = TJT^{-1}$, where

$$T = 2\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Determine $\lim\limits_{t\to\infty} y(t)$.

2. Given the autonomous differential system

$$\dot{y} = f(y),\ f \in C^1(\mathbb{R}^n), \tag{1}$$

let $y(t)$ be a continuous $n \times 1$ column vector solution of (1) on the interval (a, b). Prove the following:

(i) If $b = \infty$ and $\lim\limits_{t\to\infty} y(t) = L$ for some constant vector L, then L is a critical point of (1).

(ii) Assume the derivative $\dot{y}$ is not identically zero on (a, b) and that $\lim\limits_{t\to b^+} y(t) = L$, where L is a critical point of (1). Prove that $b = \infty$.

(iii) $y(t+\omega)$ is also a vector solution of (1) on $(a+\omega, b+\omega)$.

3. (a) Estimate an interval of existence for the initial value problem

$$\dot{y}_1 = \frac{y_2}{1-y_1}, \quad \dot{y}_2 = \frac{y_1}{1+y_2}, \qquad y_1(0) = 2, \quad y_2(0) = 1. \tag{2}$$

(b) Set up an equivalent integral equation for (2) and determine the first two successive approximations for the solution.

4. Provide examples for the following:

(a) An example of an initial value problem $\dot{y} = f(y)$, $y(0) = 0$ with $f \in C(\mathbb{R})$ such that the initial value problem does not have a unique solution.

(b) An example of an initial value problem $\dot{y} = f(y)$, $y(0) = y_0$ with $f \in C(\mathbb{R})$, with solutions that do not exist on the whole $\mathbb{R}$

(c) An example of a scalar linear homogeneous singular differential equation such that all solutions exist on the whole $\mathbb{R}$.

5. Discuss the stability of the zero solution for

$$\dot{x} = x(\cos y - y) - y(1+t^2)^{-1}, \quad \dot{y} = -2\sin x + y - 4x\sin y + 2x(1+t)^{-2}.$$

6. Consider the differential system

$$\dot{\mathbf{x}} = -A(t)\mathbf{x} - |\mathbf{x}|\mathbf{x}, \quad t \geq 0, \tag{3}$$

where $A : [0,\infty) \to \mathbb{R}^{n\times n}$ is a matrix-valued continuous function, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

(a) Assume that there exists a constant $0 < B < \infty$ such that all the entries of the matrix $A(t)$ satisfy

$$|A_{ij}(t)| \leq B \text{ for all } t \in [0,\infty) \text{ and all } 1 \leq i,\ j \leq n.$$

Prove that for any $\mathbf{x}_0 \in \mathbb{R}^n$ the initial value problem given by (3) and $\mathbf{x}(0) = \mathbf{x}_0$ has a unique solution defined on $[0,\infty)$.

(b) Assume further that there exists a continuous function $\lambda : [0,\infty) \to (0,\infty)$ such that

$$\int_0^\infty \lambda(t)dt = \infty$$

and

$$\mathbf{v}^T A(t)\mathbf{v} \geq \lambda(t)|\mathbf{v}|^2 \text{ for all } t \in [0,\infty) \text{ and all column vectors } \mathbf{v} \in \mathbb{R}^n,$$

where $\mathbf{v}^T$ and $|\mathbf{v}|$ denote the transpose and the Euclidean norm of $\mathbf{v}$, respectively. Prove that the zero solution of (3) is globally asymptotically stable.

7. Consider a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the scalar differential equation

$$\dot{x} = f(t, x) \tag{4}$$

has the following property: for all $t_0,\ x_0 \in \mathbb{R}$, the initial value problem (4) together with $x(t_0) = x_0$ has at least one solution defined on the whole $\mathbb{R}$.

Prove the following statement: for any $c_1,\ c_2 \in \mathbb{R}$ and any solutions $x_1,\ x_2$ of (4) the function $c_1x_1 + c_2x_2$ is also a solution of (4) if and only if there exists a continuous function $a : \mathbb{R} \to \mathbb{R}$ such that $f(t, x) = a(t)x$ for all $t,\ x \in \mathbb{R}$.

8. Consider the system of scalar differential equations

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y),$$

where f, g are continuous real-valued functions defined on $\mathbb{R}^2$ for which there exists a *nonnegative* real-valued function h such that

$$x[2f(x, y)+g(x, y)]+y[f(x, y)+2g(x, y)] = h(x, y)(1-x^2-y^2) \text{ for all } x,\ y \in \mathbb{R}.$$

Prove that the system has periodic solutions.