Fall 2014

Differential Equations

WVU Mathematics Department

PhD Entrance Exam, Differential Equations; Fall 2014
Solve any six problems!

1. Let $y^T = (y_1, y_2, ..., y_n)$ denote a row vector that is the transpose of $y \in \mathbb{R}^n$. In particular let $\hat{0}^T = (0, ..., 0)$ be the transpose of the zero vector. Let $f^T(y,t) := (f_1(y,t), f_2(y,t),...,f_n(y,t))$ be a vector field in $\mathbb{R}^n$ where $f_k(y,t) \in \mathbb{R}$, $k = 1, 2, ..., n$ and $t \in [a,b]$. Formulate (without proof) an existence and uniqueness theorem that will ensure that the initial value problem

$$\frac{dy}{dt}(t) = f(y,t),\ y(t_0) = \eta,\ t_0 \in [a,b], \tag{1}$$

possesses a unique solution on some subinterval of $[a,b]$. Based on this formulation prove that if

$$|f(y,t)| \leq A\,|y| + B, \quad y \in \mathbb{R}^n, t \in [a,b],$$

with some nonnegative constants A and B, then the initial value problem (1) possesses a unique solution on $[t_0, b]$.

2. (a) Determine the value of all constants c that will guarantee the existence of a bounded non trivial (not zero) vector solution $y(t)$ on $(-\infty, \infty)$ to the linear system with constant coefficients

$$y' = \begin{bmatrix} 1 & c \\ 3 & 2 \end{bmatrix} y.$$

(b) Let $A = A^T$ be a constant symmetric $n \times n$ matrix with real entries. Prove that the system

$$\frac{dy}{dt}(t) = Ay(t),$$

possesses a bounded non trivial (non zero) vector solution $y(t)$ on $(-\infty, \infty)$ iff the matrix A has a zero eigenvalue. *Hint: Use properties of symmetric matrices.*

3. Prove that the equation below has periodic solutions

$$\ddot{x} + \left(\dot{x}\right)^5 + x^2\dot{x} - \dot{x} + x^3 = 0.$$

4. Solve the initial value problem

$$\dot{y} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} y - \begin{bmatrix} 4 \\ 4 \\ 10 \end{bmatrix}, \quad y(6) = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}.$$

5. Provide examples for the following:

(a) An initial value problem $\dot{y} = f(y)$, $y(0) = 0$ with $f(y) \in C(\mathbb{R})$ such that the initial value problem does not have a unique solution.

(b) An initial value problem $\dot{y} = f(y)$, $y(0) = y_0$ with $f(y) \in C(\mathbb{R})$, with solutions that do not exist on $\mathbb{R}$.

(c) A scalar singular differential equation such that non trivial solutions exist on $\mathbb{R}$.

6. Consider the system

$$\dot{x} = y + x^2 y, \quad \dot{y} = -4x + x^3 - xy^2.$$

(a) Show that the system is Hamiltonian and find its Hamiltonian function.

(b) Find the critical points and discuss their stability.

(c) Sketch the phase diagram (make sure to put arrows to indicate the direction of motion along orbits).

7. Consider the differential system

$$\dot{x} = x^3 + x^2 y, \quad \dot{y} = y^3 + y^2 z, \quad \dot{z} = z^3 + z^2 x.$$

(a) Show that all solutions exist for all negative times;

(b) Show that all solutions converge to the origin as $t \to -\infty$;

(c) Show that with the exception of the stationary solutions (find them!), all other solutions blow up in finite time;

(d) Study the stability of the critical points.

8. (a) Discuss the stability of the null solution for the system:

$$\dot{x} = x - e^{-t} y, \quad \dot{y} = y - z(1 + e^{-t}), \quad \dot{z} = 2z + xyz^2.$$

(b) Is the origin asymptotically stable for the system:

$$\dot{x} = x^2 - y^2 z, \quad \dot{y} = xy^3 - z, \quad \dot{z} = -2xy \ ?$$

Hint for (b): Study the evolution of a point in a certain octant.