Spring 2020

# Algebra

WVU Mathematics Department

West Virginia University
DEPARTMENT OF MATHEMATICS

# M.S. Advanced/Ph.D. Entrance exam in Algebra

## April 2020

| Part | A | | | B | | | C | | | Total Score |
|---|---|---|---|---|---|---|---|---|---|---|
| # | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| √ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | |
| Pages | | | | | | | | | | |
| Score | | | | | | | | | | |

**PLEASE READ THE DIRECTIONS CAREFULLY:**

This exam has three parts:
**Part A**: Group Theory, **Part B**: Field and Galois Theory, **Part C**: Ring and Module Theory.

** *SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B AND C* **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded**.
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit.
**Do not submit scratchworks and solutions that are not to be graded**.
Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences using correct English grammar. *Answers, even correct, without justifications will not receive full credit*.
- **If you make use of a result which you do not prove, write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

## Part A. Group Theory

**Conventions.** For a given group $G$,

- $[G:H]$ denotes the *index* of a subgroup $H$ of $G$ in $G$.
- $G$ is called simple if $G \neq \{e\}$ and the only normal subgroups of $G$ are $\{e\}$ and $G$.

**Questions.**

(1) Let $G$ be a *finite* group and let $\alpha$ be an *automorphism* of $G$.
Assume $\alpha$ fixes *more than half of the elements* of $G$.
Prove that $\alpha$ is the identity map.

(Hint: The map $\alpha$ is said to fix an element $z$ of $G$ provided that $\alpha(z) = z$. Prove, and use the fact, that the set $H$ of all elements of $G$ that are fixed by $\alpha$ is a subgroup of $G$.) □

(2) Let $G$ be an *infinite* group.
Assume there is a subgroup $H$ of $G$ such that $H \neq G$ and $[G:H] < \infty$.
Prove that $G$ is *not* simple.

(Hint: Consider an action of $G$ on the set of all left cosets of $H$ in $G$.) □

(3) Let $G$ be a group of order 30.
Assume $P$ is a Sylow 3-subgroup of $G$, and $K$ is a Sylow 5-subgroup of $G$.

(a) Prove that $PK$ is a normal subgroup of $G$, and $K$ is the *unique* Sylow 5-subgroup of $PK$.
(b) If $L$ is a Sylow 5-subgroup of $G$, prove that $L$ is also a Sylow 5-subgroup of $PK$.
(c) Prove, by using parts (a) and (b), that $K$ is the *unique* Sylow 5-subgroup of $G$. □

## Part B. Field and Galois Theory

**Conventions.**

- $\mathbb{Q}$ denotes the set of rational numbers.
- A *Galois* extension is a field extension that is finite, normal, and separable.

**Questions.**

(4) Let $F \subseteq K \subseteq E$ be field extensions, where $F \neq K$, and $E = F(\alpha)$ for some $\alpha \in E$.
Prove that $E/K$ is an *algebraic* extension.

(Hint: recall that $F(\alpha)$ is the field of fractions of $F[\alpha]$.) ☐

(5) Consider the field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$, where $\alpha = \sqrt{2+\sqrt{2}}$.
Determine whether or not this field extension is *normal*. Explain your argument in detail. ☐

(6) Let $F = \mathbb{Q}$, $p(x) = x^4 + 4x^2 + 2 \in F[x]$, and let $K$ be the *splitting field* of $p(x)$ over $F$.
Let $G = \mathrm{Gal}(K/F)$ be the Galois group of the field extension $K/F$.
Prove, *by finding a generator of $G$*, that $G$ is a cyclic group of order 4. ☐

## Part C. Ring and Module Theory

**Conventions.**

- $R$ denotes a ring (not necessarily commutative) which has *multiplicative identity* 1 such that $1 \neq 0$. Moreover, all $R$-modules are assumed to be left modules.

- A ring $R$ is called a *division ring* if each nonzero element is a unit in $R$.

- A (left) module $M$ over a ring $R$ is called *simple* if $M \neq 0$ and the only (left) $R$-submodules of $M$ are 0 and $M$.

**Questions.**

(7) Let $R$ be a ring (not necessarily commutative). Assume $x^2 = x$ for all $x \in R$.
Prove that $R$ is commutative.

(8) Let $R$ be a commutative ring and let $I$ be an ideal of $R$.
Let $R[x]$ denote the polynomial ring in the variable $x$ over the ring $R$ and
let $I[x]$ denote the set of all polynomials in $R[x]$ whose coefficients belong to $I$
(Note that $I[x]$ is an ideal of $R[x]$, but you do not need to prove this.)

Determine *whether or not* the following statements are correct; if true, then provide a proof; if false, then give a counterexample and justify it.

(a) If $I$ is a *prime* ideal of $R$, then $I[x]$ is a *prime* ideal of $R[x]$.
(b) If $I$ is a *maximal* ideal of $R$, then $I[x]$ is a *maximal* ideal of $R[x]$.

(9) Let $R$ be a ring (not necessarily commutative) and let $M$ be a *simple* (left) $R$-module.
Prove that the endomorphism ring $\mathrm{End}_R(M)$ is a *division ring*.

(Hint: recall that $\mathrm{End}_R(M)$, the set of all left $R$-module homomorphisms from $M$ to $M$, is a ring with addition and composition of functions.)

Write **big** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page # of problem # | name: