

Spring 2020

## Algebra

WVU Mathematics Department



M.S. Advanced/Ph.D. Entrance exam in Algebra

April 2020

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
Pages										
Score										

**PLEASE READ THE DIRECTIONS CAREFULLY:**

This exam has three parts:

**Part A:** Group Theory, **Part B:** Field and Galois Theory, **Part C:** Ring and Module Theory.

**\*\* SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B AND C \*\***

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded.***
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit. **Do not submit scratchworks and solutions that are not to be graded.** Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences using correct English grammar. *Answers, even correct, without justifications will not receive full credit.*
- **If you make use of a result which you do not prove, write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

## Part A. Group Theory

**Conventions.** For a given group  $G$ ,

- $[G : H]$  denotes the *index* of a subgroup  $H$  of  $G$  in  $G$ .
- $G$  is called *simple* if  $G \neq \{e\}$  and the only normal subgroups of  $G$  are  $\{e\}$  and  $G$ .

**Questions.**

- (1) Let  $G$  be a *finite* group and let  $\alpha$  be an *automorphism* of  $G$ .  
Assume  $\alpha$  fixes *more than half* of the elements of  $G$ .  
Prove that  $\alpha$  is the identity map.

(Hint: The map  $\alpha$  is said to fix an element  $z$  of  $G$  provided that  $\alpha(z) = z$ . Prove, and use the fact, that the set  $H$  of all elements of  $G$  that are fixed by  $\alpha$  is a subgroup of  $G$ .) □

- (2) Let  $G$  be an *infinite* group.  
Assume there is a subgroup  $H$  of  $G$  such that  $H \neq G$  and  $[G : H] < \infty$ .  
Prove that  $G$  is *not* simple.

(Hint: Consider an action of  $G$  on the set of all left cosets of  $H$  in  $G$ .) □

- (3) Let  $G$  be a group of order 30.  
Assume  $P$  is a Sylow 3-subgroup of  $G$ , and  $K$  is a Sylow 5-subgroup of  $G$ .
- Prove that  $PK$  is a normal subgroup of  $G$ , and  $K$  is the *unique* Sylow 5-subgroup of  $PK$ .
  - If  $L$  is a Sylow 5-subgroup of  $G$ , prove that  $L$  is also a Sylow 5-subgroup of  $PK$ .
  - Prove, by using parts (a) and (b), that  $K$  is the *unique* Sylow 5-subgroup of  $G$ . □

## Part B. Field and Galois Theory

### Conventions.

- $\mathbb{Q}$  denotes the set of rational numbers.
- A *Galois* extension is a field extension that is finite, normal, and separable.

### Questions.

- (4) Let  $F \subseteq K \subseteq E$  be field extensions, where  $F \neq K$ , and  $E = F(\alpha)$  for some  $\alpha \in E$ . Prove that  $E/K$  is an *algebraic* extension.

(Hint: recall that  $F(\alpha)$  is the field of fractions of  $F[\alpha]$ .)

□

- (5) Consider the field extension  $\mathbb{Q}(\alpha)/\mathbb{Q}$ , where  $\alpha = \sqrt{2 + \sqrt{2}}$ .

Determine whether or not this field extension is *normal*. Explain your argument in detail.

□

- (6) Let  $F = \mathbb{Q}$ ,  $p(x) = x^4 + 4x^2 + 2 \in F[x]$ , and let  $K$  be the *splitting field* of  $p(x)$  over  $F$ . Let  $G = \text{Gal}(K/F)$  be the Galois group of the field extension  $K/F$ .

Prove, by *finding a generator* of  $G$ , that  $G$  is a cyclic group of order 4.

□

## Part C. Ring and Module Theory

### Conventions.

- $R$  denotes a ring (not necessarily commutative) which has *multiplicative identity*  $1$  such that  $1 \neq 0$ . Moreover, all  $R$ -modules are assumed to be left modules.
- A ring  $R$  is called a *division ring* if each nonzero element is a unit in  $R$ .
- A (left) module  $M$  over a ring  $R$  is called *simple* if  $M \neq 0$  and the only (left)  $R$ -submodules of  $M$  are  $0$  and  $M$ .

### Questions.

(7) Let  $R$  be a ring (not necessarily commutative). Assume  $x^2 = x$  for all  $x \in R$ . Prove that  $R$  is commutative.

(8) Let  $R$  be a commutative ring and let  $I$  be an ideal of  $R$ .

Let  $R[x]$  denote the polynomial ring in the variable  $x$  over the ring  $R$  and let  $I[x]$  denote the set of all polynomials in  $R[x]$  whose coefficients belong to  $I$  (Note that  $I[x]$  is an ideal of  $R[x]$ , but you do not need to prove this.)

Determine *whether or not* the following statements are correct; if true, then provide a proof; if false, then give a counterexample and justify it.

- (a) If  $I$  is a *prime* ideal of  $R$ , then  $I[x]$  is a *prime* ideal of  $R[x]$ .
- (b) If  $I$  is a *maximal* ideal of  $R$ , then  $I[x]$  is a *maximal* ideal of  $R[x]$ .

(9) Let  $R$  be a ring (not necessarily commutative) and let  $M$  be a *simple* (left)  $R$ -module. Prove that the endomorphism ring  $\text{End}_R(M)$  is a *division ring*.

(Hint: recall that  $\text{End}_R(M)$ , the set of all left  $R$ -module homomorphisms from  $M$  to  $M$ , is a ring with addition and composition of functions.)

Write **big** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page #

of problem #

name:

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