Spring 2020

Differential Equations

WVU Mathematics Department

ODE ENTRANCE EXAM, SPRING 2020

April 22 2020

Solve all six problems. Show all work. Explain and justify your answers. All problems carry equal weight.

Name ______________________ **Total Score** ________

1. Let $t_0 \in \mathbb{R}$, $\alpha > 0$, and $f : [t_0 - \alpha, t_0 + \alpha] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Suppose also that there exist nonnegative constants A, B such that

$$|f(t, y)| \leq A\,|y| + B \tag{1}$$

for all $(t, y) \in [t_0 - \alpha, t_0 + \alpha] \times \mathbb{R}^n$.

a) Prove (utilizing a proper existence and uniqueness theorem) that for any $\eta \in \mathbb{R}^n$, the initial value problem

$$y' = f(t, y), \quad y(t_0) = \eta \tag{2}$$

possesses a solution for $|t - t_0| \leq \alpha$. Provide an example that shows that a solution to (2) need not be unique.

b) How is this theorem relevant to the initial value problem

$$y' = A(t)y + g(t),\ y(t_0) = \eta\,?$$

c) Utilize a) to determine a lower bound for the maximal interval of existence for the solutions to the initial value problem

$$y' = \begin{bmatrix} \frac{1}{t^2} & 5 \\ e^t & \frac{1}{(1-t)^2} \end{bmatrix} y,\ y(0.5) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3}$$

2. Suppose that the entries of the n by n matrix $A(t) = (a_{j,k}(t))$ are continuous on the interval (a, b). Denote by $\Phi(t)$ an n by n matrix solution to

$$\Phi' = A(t)\Phi.$$

a) Prove that

$$D' = \left(\sum_{j=1}^{n} a_{j,j}(t) \right) D, \tag{4}$$

where $D(t)$ is the determinant of $\Phi(t)$.

b) Explain the significance of the solutions of the initial value problems of (4) to the theory of linear homogeneous differential systems.

3. Let $x(t)$ be a solution of the equation

$$t\frac{dx}{dt} = (2t^2 + 1)x + t^2$$

such that $\lim_{t\to\infty} x(t)$ is finite. Find this limit.

4. Consider the ODE system

$$\begin{aligned} \frac{dx}{dt} &= -2x + y \\ \frac{dy}{dt} &= -y + 2x. \end{aligned} \tag{5}$$

a) Find $a, b \in \mathbb{R}$ such that $ax(t) + by(t)$ is a constant function of t for any solution $(x(t), y(t))$ of (5). ‘

b) What are the critical points of the system? Draw a phase portrait, sketching the vector field and typical trajectories.

c) Find $\lim_{t\to\infty}(x(t), y(t))$ for the solution of (5) defined on $(-\infty, \infty)$ and satisfying $(x(0), y(0)) = (-1, 2)$.

5. Let $a : [0, \infty) \to \mathbb{R}$ be a continuous function. Show that the zero solution of

$$\frac{dx}{dt} = a(t)x$$

is asymptotically stable if and only if $\int_{t_0}^{\infty} a(s)\mathrm{d}s = -\infty$ for any $t_0 \geq 0$.

6. Given an autonomous vector system

$$y' = f(y),\ f \in C(\mathbb{R}^n), \tag{6}$$

prove that if $y(t)$ is a vector solution of (6) such that $L = \lim_{t\to\infty} y(t)$ exists and is finite, then L is a critical point of (6).