

Spring 2020

Differential Equations

WVU Mathematics Department

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ODE ENTRANCE EXAM, SPRING 2020

April 22 2020

Solve all six problems. Show all work. Explain and justify your answers. All problems carry equal weight.

Name _____ Total Score _____

1. Let $t_0 \in \mathbb{R}$, $\alpha > 0$, and $f : [t_0 - \alpha, t_0 + \alpha] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous. Suppose also that there exist nonnegative constants A, B such that

$$|f(t, y)| \leq A|y| + B \quad (1)$$

for all $(t, y) \in [t_0 - \alpha, t_0 + \alpha] \times \mathbb{R}^n$.

a) Prove (utilizing a proper existence and uniqueness theorem) that for any $\eta \in \mathbb{R}^n$, the initial value problem

$$y' = f(t, y), \quad y(t_0) = \eta \quad (2)$$

possesses a solution for $|t - t_0| \leq \alpha$. Provide an example that shows that a solution to (2) need not be unique.

b) How is this theorem relevant to the initial value problem

$$y' = A(t)y + g(t), \quad y(t_0) = \eta ?$$

c) Utilize a) to determine a lower bound for the maximal interval of existence for the solutions to the initial value problem

$$y' = \begin{bmatrix} \frac{1}{t^2} & 5 \\ e^t & \frac{1}{(1-t)^2} \end{bmatrix} y, \quad y(0.5) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (3)$$

2. Suppose that the entries of the n by n matrix $A(t) = (a_{j,k}(t))$ are continuous on the interval (a, b) . Denote by $\Phi(t)$ an n by n matrix solution to

$$\Phi' = A(t)\Phi.$$

a) Prove that

$$D' = \left(\sum_{j=1}^n a_{j,j}(t) \right) D, \quad (4)$$

where $D(t)$ is the determinant of $\Phi(t)$.

b) Explain the significance of the solutions of the initial value problems of (4) to the theory of linear homogeneous differential systems.

3. Let $x(t)$ be a solution of the equation

$$t \frac{dx}{dt} = (2t^2 + 1)x + t^2$$

such that $\lim_{t \rightarrow \infty} x(t)$ is finite. Find this limit.

4. Consider the ODE system

$$\begin{aligned} \frac{dx}{dt} &= -2x + y \\ \frac{dy}{dt} &= -y + 2x. \end{aligned} \quad (5)$$

a) Find $a, b \in \mathbb{R}$ such that $ax(t) + by(t)$ is a constant function of t for any solution $(x(t), y(t))$ of (5). ‘

b) What are the critical points of the system? Draw a phase portrait, sketching the vector field and typical trajectories.

c) Find $\lim_{t \rightarrow \infty} (x(t), y(t))$ for the solution of (5) defined on $(-\infty, \infty)$ and satisfying $(x(0), y(0)) = (-1, 2)$.

5. Let $a : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. Show that the zero solution of

$$\frac{dx}{dt} = a(t)x$$

is asymptotically stable if and only if $\int_{t_0}^{\infty} a(s)ds = -\infty$ for any $t_0 \geq 0$.

6. Given an autonomous vector system

$$y' = f(y), \quad f \in C(\mathbb{R}^n), \quad (6)$$

prove that if $y(t)$ is a vector solution of (6) such that $L = \lim_{t \rightarrow \infty} y(t)$ exists and is finite, then L is a critical point of (6).