Spring 2020

Topology

WVU Mathematics Department

NAME (print): ____________________

Topology Ph.D. Entrance Exam, April 2020

Solve ***four*** from the following five exercises. Write a solution of each exercise on a separate page. After you finish, you will need to take pictures of your work and sent it to the examiners according to the separate instructions.

In what follows the symbols $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ stand, respectively, for the interior and the closure of A. Any subset of $\mathbb{R}$ is considered with the standard topology, unless stated otherwise.

Ex. 1. Let A be a connected subset of a topological space X and let $A \subset B \subset \mathrm{cl}(A)$. Show that B is a connected subset of X.

Ex. 2. Let X be an arbitrary set, $\langle Y, \tau \rangle$ a topological space, and $f\colon X \to Y$ an arbitrary function. Show that

1. $\mathcal{T}_0 = \{f^{-1}(V)\colon V \in \tau\}$ is a topology on X.
2. If $\mathcal{T}$ is any topology on X such that $f\colon \langle X, \mathcal{T} \rangle \to \langle Y, \tau \rangle$ is continuous, then $\mathcal{T}_0 \subseteq \mathcal{T}$.

Ex. 3. Let $\langle X, \rho \rangle$ be a compact metric space.

Show that every sequence in X contains a subsequence converging to an $x \in X$.

Ex. 4. Determine which of the following families of subsets of $\mathbb{R}$ form a basis for some topology on $\mathbb{R}$. Briefly justify your answers.

(1) $\mathcal{B}_1 = \{(-\infty, b]\colon b \in (-\infty, 0]\}$

(2) $\mathcal{B}_2 = \{(-\infty, b]\colon b \in \mathbb{R}\}$

(3) $\mathcal{B}_3 = \{(-\infty, b]\colon b \in \mathbb{R}\} \cup \{(a, \infty)\colon a \in \mathbb{R}\}$

(4) $\mathcal{B}_4 = \{(-\infty, b] \cap (a, \infty)\colon a, b \in \mathbb{R}\}$

Ex. 5. For a topological space a family $\mathcal{F}$ of subsets of X (not necessarily open) is a *network* for X provided that for every open subset U of X and $x \in U$ there exists an $F \in \mathcal{F}$ such that $x \in F \subset U$.

Let X be a regular topological space. Prove that if $\mathcal{F}$ is a network for X then so is the family $\mathcal{G} = \{\mathrm{cl}(F)\colon F \in \mathcal{F}\}$, where $\mathrm{cl}(F)$ is the closure of F.