Fall 2020

Topology

WVU Mathematics Department

NAME (print): ______________________________

Topology Ph.D. Entrance Exam, September 2020

Solve ***four*** from the following five exercises. *We will grade only four of them, so you need to decide which four solutions to submit.* Write a solution of each exercise on a separate page. After you finish, you will need to take pictures of your work and sent it to the examiners according to the separate instructions.

In what follows the symbol $\mathrm{cl}(A)$ stands for the closure of A. Any subset of $\mathbb{R}$ is considered with the standard topology, unless stated otherwise.

Ex. 1. For $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ let $X_n = \{0, 1, \ldots, n-1\}$.

(a) Show that for any $n \geq 2$ any Hausdorff topology on X_n is discrete.

(b) Show that for any $n \geq 2$ there exists a topology $\mathcal{T}_n$ on X_n that is neither trivial (i.e., equal to $\{\emptyset, X_n\}$) nor discrete. Explain why your family $\mathcal{T}_n$ is indeed a topology.

Ex. 2. Let $\langle X, d\rangle$ be a metric space. For nonempty sets $A, B \subset X$ define

$$d(A, B) = \inf\{d(a, b) \colon a \in A \;\&\; b \in B\}.$$

(a) Show that $d(\mathrm{cl}(A), \mathrm{cl}(B)) = d(A, B)$ for every nonempty $A, B \subset X$.

(b) Give an example of a metric space $\langle X, d\rangle$ and nonempty closed disjoint $A, B \subset X$ with $d(A, B) = 0$.

(c) Prove that an example as in the part (b) does not exist when we require $\langle X, d\rangle$ to be also compact.

Ex. 3. Let $\langle A_n \colon n = 1, 2, 3, \ldots\rangle$ be a sequence of nonempty subsets of a topological space X such that $A_n \cup A_{n+1}$ is connected for every n.

(a) Show that $A := \bigcup_{n=1}^{\infty} A_n$ is a connected subset of X.

(b) Give an example of a metric space X and a sequence $\langle A_n \colon n = 1, 2, 3, \ldots\rangle$ such that $\bigcup_{n=1}^{\infty} A_n$ is connected, but some of the sets A_n are disconnected.

Ex. 4. Let $X_n = \mathbb{R}$ for each $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and consider

$$X := \prod_{n=1}^{\infty} X_n$$

with the product topology. Let $x_0 = (0, 0, 0, \ldots) \in X$. For each finite $A \subset \mathbb{N}$ choose an element x_A of X such that the n-th coordinate of x_A is 0 for every $n \in A$. Let $Y = \{x_A \colon A \subset \mathbb{N}$ & A is finite$\}$. Prove that $x_0 \in \mathrm{cl}(Y)$.

Ex. 5. Let $\mathcal{E}$ be the family of all subsets of $\mathbb{R}$ that are unbounded from above and from below.

(a) Show that if $E \in \mathcal{E}$, then $\mathcal{B}_E := \{[a, b) \colon a < b \text{ and } a, b \in E\}$ is a basis for a topology on $\mathbb{R}$, where $[a, b) = \{c \in \mathbb{R} \colon a \leq c < b\}$. (Thus $[a, b) = \emptyset$ when $a \geq b$.)

(b) For $E \in \mathcal{E}$ let $\mathcal{T}_E$ be the topology on $\mathbb{R}$ generated by the basis $\mathcal{B}_E$. Prove that for every $D, E \in \mathcal{E}$ if $D \subsetneq E$, then $\mathcal{T}_D \subsetneq \mathcal{T}_E$.