

Fall 2020

Topology

WVU Mathematics Department

NAME (print): _____

Topology Ph.D. Entrance Exam, September 2020

Solve **four** from the following five exercises. *We will grade only four of them, so you need to decide which four solutions to submit.* Write a solution of each exercise on a separate page. After you finish, you will need to take pictures of your work and sent it to the examiners according to the separate instructions.

In what follows the symbol $\text{cl}(A)$ stands for the closure of A . Any subset of \mathbb{R} is considered with the standard topology, unless stated otherwise.

Ex. 1. For $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ let $X_n = \{0, 1, \dots, n-1\}$.

- (a) Show that for any $n \geq 2$ any Hausdorff topology on X_n is discrete.
- (b) Show that for any $n \geq 2$ there exists a topology \mathcal{T}_n on X_n that is neither trivial (i.e., equal to $\{\emptyset, X_n\}$) nor discrete. Explain why your family \mathcal{T}_n is indeed a topology.

Ex. 2. Let $\langle X, d \rangle$ be a metric space. For nonempty sets $A, B \subset X$ define

$$d(A, B) = \inf\{d(a, b) : a \in A \ \& \ b \in B\}.$$

- (a) Show that $d(\text{cl}(A), \text{cl}(B)) = d(A, B)$ for every nonempty $A, B \subset X$.
- (b) Give an example of a metric space $\langle X, d \rangle$ and nonempty closed disjoint $A, B \subset X$ with $d(A, B) = 0$.
- (c) Prove that an example as in the part (b) does not exist when we require $\langle X, d \rangle$ to be also compact.

Ex. 3. Let $\langle A_n : n = 1, 2, 3, \dots \rangle$ be a sequence of nonempty subsets of a topological space X such that $A_n \cup A_{n+1}$ is connected for every n .

- (a) Show that $A := \bigcup_{n=1}^{\infty} A_n$ is a connected subset of X .
- (b) Give an example of a metric space X and a sequence $\langle A_n : n = 1, 2, 3, \dots \rangle$ such that $\bigcup_{n=1}^{\infty} A_n$ is connected, but some of the sets A_n are disconnected.

Ex. 4. Let $X_n = \mathbb{R}$ for each $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ and consider

$$X := \prod_{n=1}^{\infty} X_n$$

with the product topology. Let $x_0 = (0, 0, 0, \dots) \in X$. For each finite $A \subset \mathbb{N}$ choose an element x_A of X such that the n -th coordinate of x_A is 0 for every $n \in A$. Let $Y = \{x_A : A \subset \mathbb{N} \text{ \& } A \text{ is finite}\}$. Prove that $x_0 \in \text{cl}(Y)$.

Ex. 5. Let \mathcal{E} be the family of all subsets of \mathbb{R} that are unbounded from above and from below.

- (a) Show that if $E \in \mathcal{E}$, then $\mathcal{B}_E := \{[a, b) : a < b \text{ and } a, b \in E\}$ is a basis for a topology on \mathbb{R} , where $[a, b) = \{c \in \mathbb{R} : a \leq c < b\}$. (Thus $[a, b) = \emptyset$ when $a \geq b$.)
- (b) For $E \in \mathcal{E}$ let \mathcal{T}_E be the topology on \mathbb{R} generated by the basis \mathcal{B}_E . Prove that for every $D, E \in \mathcal{E}$ if $D \subsetneq E$, then $\mathcal{T}_D \subsetneq \mathcal{T}_E$.