

Fall 2020

Real Analysis

WVU Mathematics Department

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Entrance Exam, Real Analysis

September 16, 2020

Solve exactly 6 out of the 8 problems. Circle the problem numbers to be graded.

1. Prove by definition

(a) function $f(x) = x^2$ is continuous at $x = -3$;

(b) function $f(x) = x^2$ is not uniformly continuous in $(-3, +\infty)$.

2. Does

$$f_n(x) = x^n \sin\left(\frac{1-x}{x}\right)$$

converge uniformly on $(0,1)$? Prove your conclusion.

3. Evaluate the integral

$$I = \int_0^1 dy \int_y^1 e^{-x^2} dx.$$

4. Answer the following questions and prove your conclusion.

(a) Let $A \subset [0, 1]$ and A be dense in $[0, 1]$. If $mA < 1/2$, is it possible that A is open?

(b) Let $B \subset [0, 1]$ and $B \neq [0, 1]$. If $mB = 1$, is it possible that B is closed?

5. Given a bounded function $f : [a, b] \rightarrow \mathbb{R}$, let

$$H(x) = \lim_{\delta \rightarrow 0} \sup_{|y-x| < \delta} f(y), \quad h(x) = \lim_{\delta \rightarrow 0} \inf_{|y-x| < \delta} f(y).$$

Show that $H(x) = h(x)$ if and only if f is continuous at $a < x < b$.

6. Show the following: If $f_n \rightarrow f$ almost everywhere on \mathbb{R} and $|f_n| \leq g$ for all n , where g is Lebesgue integrable, then $\int |f_n - f| dx \rightarrow 0$.

7. Consider the following functions on \mathbb{R} .

(a) $f_n = n^{-1}\chi_{(0,n)}$, (b) $f_n = \chi_{(n,n+1)}$, (c) $f_n = n\chi_{[0,1/n]}$,

where

$$\chi_{(a,b]}(x) = \begin{cases} 0 & x \leq a \\ 1 & a < x \leq b \\ 0 & b < x \end{cases}.$$

Consider three modes of convergence for $f_n \rightarrow 0$, *i.e.*, uniformly, pointwise, and almost everywhere. State which apply to (a), (b), (c). List all that apply.

8. Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$. Show the following.

(a) $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x) dx = 0$

(b) $\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) dx = \ln(b/a)$

(c) $\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(x)| dx = \infty$

(d) Explain why (a) and (b) differ.