Fall 2020

Real Analysis

WVU Mathematics Department

Entrance Exam, Real Analysis

September 16, 2020

Solve exactly 6 out of the 8 problems. Circle the problem numbers to be graded.

1. Prove by definition
   (a) function $f(x) = x^2$ is continuous at $x = -3$;
   (b) function $f(x) = x^2$ is not uniformly continuous in $(-3, +\infty)$.

2. Does
$$f_n(x) = x^n \sin\left(\frac{1-x}{x}\right)$$
converge uniformly on (0,1)? Prove your conclusion.

3. Evaluate the integral
$$I = \int_0^1 dy \int_y^1 e^{-x^2} dx.$$

4. Answer the following questions and prove your conclusion.
   (a) Let $A \subset [0,1]$ and A be dense in $[0,1]$. If $mA < 1/2$, is it possible that A is open?
   (b) Let $B \subset [0,1]$ and $B \neq [0,1]$. If $mB = 1$, is it possible that B is closed?

5. Given a bounded function $f : [a,b] \to R$, let
$$H(x) = \lim_{\delta\to 0} \sup_{|y-x|<\delta} f(y), \qquad h(x) = \lim_{\delta\to 0} \inf_{|y-x|<\delta} f(y).$$
Show that $H(x) = h(x)$ if and only if f is continuous at $a < x < b$.

6. Show the following: If $f_n \to f$ almost everywhere on R and $|f_n| \leq g$ for all n, where g is Lebesgue integrable, then $\int |f_n \to f| dx \to 0$.

7. Consider the following functions on R.
   (a) $f_n = n^{-1}\chi_{(0,n)}$, (b) $f_n = \chi_{(n,n+1)}$, (c) $f_n = n\chi_{[0,1/n]}$,
where
$$\chi_{(a,b]}(x) = \begin{cases} 0 & x \leq a \\ 1 & a < x \leq b \\ 0 & b < x \end{cases}.$$
Consider three modes of convergence for $f_n \to 0$, *i.e.*, uniformly, pointwise, and almost everywhere. State which apply to (a), (b), (c). List all that apply.

8. Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$. Show the following.
   (a) $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)dx = 0$
   (b) $\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)dx = \ln(b/a)$
   (c) $\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(x)|dx = \infty$
   (d) Explain why (a) and (b) differ.