M.S. Advanced and Ph.D. Entrance Exams | Mathematics

Fall 2020

Differential Equations

WVU Mathematics Department

ODE ENTRANCE EXAM, FALL 2020

September 16 2020

Solve all six problems. Show all work. Explain and justify your answers. All problems carry equal weight.
Notation: $y' = \frac{dy}{dt}$

Name ____ **Total Score** ________

1. Let $m, n \in \mathbb{N}$ and consider the differential system

$$x' = my^{2m-1},\ y' = -nx^{2n-1}. \tag{1}$$

(x, y are scalar functions of t). Show that $(0,0)$ is a stable equilibrium of (1).

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(y) = \begin{bmatrix} \cos(y_1 y_2 y_3) \\ \sin(y_1 + y_2 + y_3) \\ \exp\left(\dfrac{2y_1 y_2}{1 + y_1^2 + y_2^2}\right) \end{bmatrix}$$

for all $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$. Show that for the differential system

$$y' = f(y)$$

a) there are no critical points;
b) all solutions exist on $(-\infty, \infty)$.

3. Let $n > 1$ be a natural number.
a) Let A be an $n \times n$ constant matrix. Prove that $Y(t) = \exp(tA)$ is a solution to the matrix differential equation $Y' = AY$.
b) Let $A(t)$ be an $n \times n$ time dependent matrix function of t on $(-\infty, \infty)$. Must $Y(t) = \exp(tA(t))$ be a solution of $Y' = A(t)Y$? Explain.

4. Solve the differential equation

$$y' = t^2 y - 3t^2$$

5. Consider the ODE system

$$x' = y^3 - 4x, \; y' = y^3 - y - 3x.$$

a) Find all critical points and classify their stability.
b) Show that the line $x = y$ is invariant (any solution that starts on it stays on it).
c) If $(x(0), y(0)) = (-1, 1)$, find $x(t) - y(t)$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(t) > f(t)$ for all $t \in \mathbb{R}$, and $f(0) = 0$. Show that $f(t) > 0$ for any $t > 0$.