Fall 2020

Algebra

WVU Mathematics Department

M.S. Advanced/Ph.D. Entrance exam in Algebra

September 2020

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	☐	☐	☐	☐	☐	☐	☐	☐	☐	
Pages										
Score										

PLEASE READ THE DIRECTIONS CAREFULLY:

This exam has three parts:
Part A: Group Theory, **Part B**: Field and Galois Theory, **Part C**: Ring and Module Theory.

** *SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B AND C* **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded**.
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit.
Do not submit scratchworks and solutions that are not to be graded.
Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences using correct English grammar. *Answers, even correct, without justifications will not receive full credit*.
- **If you make use of a result which you do not prove, write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

Part A. Group Theory

Conventions.

• If G is a finite group, then $|G|$ denotes the order of G.

• A group G is called *simple* if $G \neq \{e\}$ and the only normal subgroups of G are $\{e\}$ and G.

• If G is a group and H is a subgroup of G, then $N_G(H)$ denotes the *normalizer* of H in G, that is, $N_G(H) = \{g \in G : gH = Hg\}$.

Questions.

(1) Let G be a group and let H be a subgroup of G. Assume the following condition holds:

$$\text{If } a, b \in G \text{ and } Ha \neq Hb, \text{ then } aH \neq bH.$$

Prove that H is a normal subgroup of G. □

(2) Let G be a finite group such that $|G| = 4 \cdot 3^s$ for some integer $s \geq 2$.
Prove that G is <u>not</u> simple. □

(3) Let G be a finite group.
Assume $|G| = p^m q$, where p and q are distinct prime numbers and m is a positive integer.
Assume further that G has a Sylow q-subgroup Q such that $\mathrm{N}_G(Q) = Q$.
Prove that G has a normal Sylow p-subgroup. □

Part B. Field and Galois Theory

Conventions.

- $\mathbb{Q}$ denotes the set of rational numbers.
- $[E:F]$ denotes the *degree* of a given field extension E/F.
- A *Galois* extension is a field extension that is finite, normal, and separable.

Questions.

(4) Let E/F be a field extension such that $[E:F]=2$.
Prove that E/F is a *normal* extension. $\square$

(5) Let K be the splitting field of the polynomial $p(x)=x^6-4$ over $\mathbb{Q}$.
Determine K and $[K:\mathbb{Q}]$. $\square$

(6) Let K/F be a *Galois* extension. Prove that there exists a field L with the following properties:

(i) $F \subseteq L \subseteq K$.
(ii) L/F is a Galois extension and $\mathrm{Gal}(L/F)$ is Abelian.
(iii) whenever E is a field such that $F \subseteq E \subseteq K$, E/F is a Galois extension, and $\mathrm{Gal}(E/F)$ is Abelian, it follows that $E \subseteq L$.

(Hint: consider the commutator subgroup of $G=\mathrm{Gal}(K/F)$ and its fixed field.) $\square$

Part C. Ring and Module Theory

Conventions.

- In this section R denotes a commutative ring which has *multiplicative identity* 1 such that $1 \neq 0$. Moreover, all R-modules are assumed to be left modules.

- Each ring homomorphism sends 1 to 1 by definition.

Questions.

(7) Let R be a ring and let $R[x]$ be the polynomial ring over R in the indeterminate x.
Prove that there is no *ring isomorphism* $f : R[x] \to \mathbb{Z}$.

(Hint: suppose such an isomorphism exists and consider the image of R under f to obtain a contradiction.) $\square$

(8) Let R be an *Artinian* ring, that is to say, whenever

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

is a *descending* chain of ideals of R, then the chain eventually stabilizes; in other words, there exists a positive integer k such that $I_k = I_{k+i}$ for all $i \geq 1$.

Prove that, if p and q are prime ideals of R such that $p \subseteq q$, then $p = q$. $\square$

(9) Let R be a ring. Assume there exists a nonzero R-module M with the following properties:
(a) M is *simple,* that is, the only R-submodules of M are 0 and M.
(b) M is *faithful,* that is, $\mathrm{Ann}_R(M) = \{r \in R : rx = 0 \text{ for all } x \in M\} = 0$.

Prove that R is a field. $\square$

Write **big** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page # of problem # | name: