

Fall 2020

Algebra

WVU Mathematics Department

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M.S. Advanced/Ph.D. Entrance exam in Algebra

September 2020

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
Pages										
Score										

PLEASE READ THE DIRECTIONS CAREFULLY:

This exam has three parts:

Part A: Group Theory, **Part B:** Field and Galois Theory, **Part C:** Ring and Module Theory.

**** SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B AND C ****

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded.***
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit. **Do not submit scratchworks and solutions that are not to be graded.** Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences using correct English grammar. *Answers, even correct, without justifications will not receive full credit.*
- **If you make use of a result which you do not prove, write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

Part A. Group Theory

Conventions.

- If G is a finite group, then $|G|$ denotes the order of G .
- A group G is called *simple* if $G \neq \{e\}$ and the only normal subgroups of G are $\{e\}$ and G .
- If G is a group and H is a subgroup of G , then $N_G(H)$ denotes the *normalizer* of H in G , that is, $N_G(H) = \{g \in G : gH = Hg\}$.

Questions.

(1) Let G be a group and let H be a subgroup of G . Assume the following condition holds:

$$\text{If } a, b \in G \text{ and } Ha \neq Hb, \text{ then } aH \neq bH.$$

Prove that H is a normal subgroup of G .

□

(2) Let G be a finite group such that $|G| = 4 \cdot 3^s$ for some integer $s \geq 2$.

Prove that G is not simple.

□

(3) Let G be a finite group.

Assume $|G| = p^m q$, where p and q are distinct prime numbers and m is a positive integer.

Assume further that G has a Sylow q -subgroup Q such that $N_G(Q) = Q$.

Prove that G has a normal Sylow p -subgroup.

□

Part B. Field and Galois Theory

Conventions.

- \mathbb{Q} denotes the set of rational numbers.
- $[E : F]$ denotes the *degree* of a given field extension E/F .
- A *Galois* extension is a field extension that is finite, normal, and separable.

Questions.

- (4) Let E/F be a field extension such that $[E : F] = 2$.
Prove that E/F is a *normal* extension.

- (5) Let K be the splitting field of the polynomial $p(x) = x^6 - 4$ over \mathbb{Q} .
Determine K and $[K : \mathbb{Q}]$.

- (6) Let K/F be a *Galois* extension. Prove that there exists a field L with the following properties:
- $F \subseteq L \subseteq K$.
 - L/F is a Galois extension and $\text{Gal}(L/F)$ is Abelian.
 - whenever E is a field such that $F \subseteq E \subseteq K$, E/F is a Galois extension, and $\text{Gal}(E/F)$ is Abelian, it follows that $E \subseteq L$.

(Hint: consider the commutator subgroup of $G = \text{Gal}(K/F)$ and its fixed field.)

Part C. Ring and Module Theory

Conventions.

- In this section R denotes a commutative ring which has *multiplicative identity* 1 such that $1 \neq 0$. Moreover, all R -modules are assumed to be left modules.
- Each ring homomorphism sends 1 to 1 by definition.

Questions.

- (7) Let R be a ring and let $R[x]$ be the polynomial ring over R in the indeterminate x . Prove that there is no ring isomorphism $f : R[x] \rightarrow \mathbb{Z}$.

(Hint: suppose such an isomorphism exists and consider the image of R under f to obtain a contradiction.) □

- (8) Let R be an *Artinian* ring, that is to say, whenever

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

is a *descending* chain of ideals of R , then the chain eventually stabilizes; in other words, there exists a positive integer k such that $I_k = I_{k+i}$ for all $i \geq 1$.

Prove that, if p and q are prime ideals of R such that $p \subseteq q$, then $p = q$. □

- (9) Let R be a ring. Assume there exists a nonzero R -module M with the following properties:
- (a) M is *simple*, that is, the only R -submodules of M are 0 and M .
 - (b) M is *faithful*, that is, $\text{Ann}_R(M) = \{r \in R : rx = 0 \text{ for all } x \in M\} = 0$.

Prove that R is a field. □

Write **big** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page #

of problem #

name:

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