Fall 2021

Algebra

WVU Mathematics Department

M.S. Advanced/Ph.D. Entrance exam in Algebra

September 2021

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	☐	☐	☐	☐	☐	☐	☐	☐	☐	
Pages										
Score										

PLEASE READ THE DIRECTIONS CAREFULLY:

This exam has three parts:
Part A: Group Theory, **Part B**: Field and Galois Theory, **Part C**: Ring and Module Theory.

** *SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B, AND C* **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded**.
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit. **Do not submit scratchworks and solutions that are not to be graded**. Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences. *Answers, even correct, without justifications will not receive (full) credit*.
- **If you make use of a result which you do not prove, then write the complete statement of the result you use separately underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

Part A. Group Theory

Questions.

(1) Let G be a nontrivial group (not necessarily finite).
Assume G has only trivial subgroups, i.e., the only subgroups of G are $\{e\}$ and G itself.
Prove that G is a finite group of order p for some prime integer p.

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(2) Let G be a finite Abelian group acting on a finite set S.
Assume the action of G on S is *faithful* and *transitive*.
Prove that the order $|G|$ of G equals the cardinality $|S|$ of S.

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(3) Let G be a *finite* group and let H be a *normal* subgroup of G.
Assume $|H| = p^i$ for some prime integer p and some integer $i \geq 1$.
Prove that H is contained in *each* Sylow p-subgroup of G.

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Part B. Field and Galois Theory

Conventions.

- A *Galois* extension is a field extension that is finite, normal, and separable.

Questions.

(4) Let $F \subsetneq K \subseteq E$ be field extensions, where $E = F(\alpha)$ for some $\alpha \in E$.
Prove that E/K is algebraic. $\big($Hint: recall that $F(\alpha)$ is the field of fractions of $F[\alpha]\big)$.

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(5) Let F be a *finite* field. If E/F is an *algebraic* field extension, prove that E/F is *normal*.

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(6) Let k be a field of prime characteristic p, and let $n \geq 1$ be an integer with p does not divide n.
Assume k contains a primitive nth root of unity, say ξ.
If $a \in k$ and α is a root of the polynomial $x^n - a \in k[x]$, then prove that $k(\alpha)/k$ is a Galois extension, the Galois group $\mathrm{Gal}(k(\alpha)/k)$ is cyclic, and the order of $\mathrm{Gal}(k(\alpha)/k)$ divides n.

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Part C. Ring and Module Theory

Conventions.

• In this section R denotes a ring (not necessarily commutative) which has *multiplicative identity* 1 such that $1 \neq 0$. Moreover, all R-modules are assumed to be left modules.

Questions.

(7) Assume R is a commutative ring.
Let $\mathfrak{p}$ be a prime ideal of R, and let $I_1, \ldots, I_n$ be ideals of R for some $n \geq 1$.
Assume $\bigcap_{i=1}^{n} I_i \subseteq \mathfrak{p}$. Prove that $I_j \subseteq \mathfrak{p}$ for some j with $1 \leq j \leq n$.

$* * *$

(8) Let M be an R-module.
Assume $M = M_1 \oplus M_2$ for some nonzero R-submodules M_1 and M_2.
Prove that there exists an R-module homomorphism $f : M \to M$ such that $f^2 = f$ and $f \neq 0, 1$, that is, f is not the zero or the identity map (here $f^2 = f \circ f$).

$* * *$

(9) Assume R is a commutative ring, and let M be a nonzero R-module.
Assume M is Noetherian. Assume further M is faithful, i.e., $0 = \mathrm{Ann}_R(M) = \{r \in R : rM = 0\}$.
Prove that R is a Noetherian ring.

$* * *$

Write **big** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page # of problem # | name: