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## A HYBRID COLLOCATION METHOD FOR VOLTERRA INTEGRAL EQUATIONS WITH WEAKLY SINGULAR KERNELS\*

YANZHAO CAO<sup>†</sup>, TERRY HERDMAN<sup>‡</sup>, AND YUESHENG XU<sup>§</sup>

**Abstract.** The commonly used graded piecewise polynomial collocation method for weakly singular Volterra integral equations may cause serious round-off error problems due to its use of extremely nonuniform partitions and the sensitivity of such time-dependent equations to round-off errors. The singularity preserving (nonpolynomial) collocation method is known to have only local convergence. To overcome the shortcoming of these well-known methods, we introduce a hybrid collocation method for solving Volterra integral equations of the second kind with weakly singular kernels. In this hybrid method we combine a singularity preserving (nonpolynomial) collocation method used near the singular point of the derivative of the solution and a graded piecewise polynomial collocation method used for the rest of the domain. We prove the optimal order of *global* convergence for this method. The convergence analysis of this method is based on a singularity expansion of the exact solution of the equations. We prove that the solutions of such equations can be decomposed into two parts, with one part being a linear combination of some known singular functions which reflect the singularity of the solutions and the other part being a smooth function. A numerical example is presented to demonstrate the effectiveness of the proposed method and to compare it to the graded collocation method.

**Key words.** Volterra integral equations, hybrid collocation methods, weakly singular kernels

**AMS subject classifications.** 65R20, 45D05

**PII.** S0036142901385593

**1. Introduction.** We propose in this paper a hybrid collocation method for solving Volterra integral equations of the second kind with weakly singular kernels. By using the singularity expansion of the exact solution, we analyze this method and prove that it has an optimal order of global convergence. Specifically, for given kernels  $K, M \in C(I \times I)$  with  $I := [0, 1]$  and a given parameter  $\alpha \in (0, 1)$ , we define a Volterra integral operator  $T_\alpha : C(I) \rightarrow C(I)$  by

$$(T_\alpha y)(t) = \int_0^t G_\alpha(t, s)y(s)ds, \quad t \in I,$$

where

$$G_\alpha(t, s) := (t - s)^{\alpha-1}K(t, s) + M(t, s) \quad \text{for } 0 \leq s \leq t, \quad 0 \leq t \leq 1,$$

and consider the Volterra integral equation of the second kind

$$(1.1) \quad y(t) - (T_\alpha y)(t) = f(t), \quad t \in I,$$

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where  $f \in C(I)$  is a given function and  $y \in C(I)$  is the unknown function to be determined. The kernel  $M$  is of practical importance because it occurs in the applications to aeroelastic modeling problems [7], where a class of neutral delay equations are converted to integral equations in the form of (1.1). Other related references include [4, 11].

Since  $0 < \alpha < 1$ , the kernel  $G_\alpha$  has a singularity along the diagonal. When (1.1) is solved by a numerical method such as a collocation method or a product-integration method using the piecewise polynomial approximation, the accuracy of the approximate solution depends on the order of piecewise polynomials used in the approximation as well as the degree of smoothness of the exact solution. For instance, when  $y \in C^r(I)$  and the approximate subspaces are chosen to be piecewise polynomials of order  $r$ , the optimal order  $r$  of convergence for the approximate solution  $y_h$  to  $y$  is achieved, that is,

$$(1.2) \quad \|y - y_h\|_\infty = O(N^{-r}),$$

where  $N$  is the number of subintervals in the uniform partition associated with the piecewise polynomial spaces. However, the solution of (1.1) exhibits, in general, singularities at the zero in its derivatives even if the forcing term  $f$  is a smooth function and the numerical methods mentioned above may not even yield first order accuracy (see, e.g., [2, 5]). In other words, the use of piecewise polynomials of high order does not produce high order convergence for the numerical method.

There have been many attempts to overcome the difficulties caused by the singularity of the solution of (1.1). One of the most commonly used methods [3, 5, 6, 9, 10, 14, 15, 16] is the graded collocation (GC) method using piecewise polynomials with a graded mesh on interval  $I$  according to the behavior of the exact solution near the singular point, which was first introduced by Rice in [13]. Specifically, the GC method partitions  $I$  by the following knots:

$$(1.3) \quad t_i = \left(\frac{i}{N}\right)^{\frac{1}{\alpha}}, \quad i = 0, 1, \dots, N,$$

which ensures that the GC method retains the optimal error estimate (1.2). However, as pointed out in [2, 10], the main disadvantage of the GC method is that subintervals near the singular point in the graded mesh have very small length and thus may cause serious round-off error problems for small  $\alpha$  and high order polynomials. Since Volterra equations are time-dependent equations, the numerical solutions of these equations are very sensitive to round-off errors.

Another approach for solving (1.1) is to include some nonpolynomial singular functions which reflect the singularity of the exact solution as part of the basis for the finite dimensional subspace in the collocation method (see [2]). We call it the nonpolynomial collocation (NPC) method. For this method, only a *local* convergence result (in [2]) has been seen so far. It does not seem that an optimal order of *global* convergence can be proved for this method. The idea of including some known singular functions in the usual finite element spaces or piecewise polynomial spaces has been explored in [8] to successfully construct Galerkin methods of high convergence order for Fredholm integral equations of the second kind with weakly singular kernels. This idea leads us to the consideration of the present method.

To treat the problems discussed above for the existing methods, we propose a hybrid collocation (HC) method for solving (1.1) which combines the strength of both the GC and NPC methods. In this method, we introduce a graded mesh different from

(1.3) that avoids using small subintervals near the zero and uses the nonpolynomial function approximation *only* in the first subinterval. Specifically, the length of the first subinterval in the HC method is the same as in a *quasi-uniform* partition, that is, there exist positive constants  $c_1, c_2$  such that

$$\frac{c_1}{N} \leq t_1 \leq \frac{c_2}{N},$$

and a graded partition is used only on  $[t_1, 1]$  so that the instability problem appearing in the GC method can be avoided. We compensate the use of a large subinterval for the first interval in the partition by employing nonpolynomial functions  $t^{i+j\alpha}$ ,  $i+j\alpha < r$ , which characterize the singularity of the exact solution  $y$  of (1.1), as trial functions in the first subinterval  $[t_0, t_1]$ . The primary purpose of this paper is to prove that this method provides an optimal order of *global* convergence by taking the strength of both the GC method and the NPC method, while avoiding the problems from which both these methods have suffered.

To prepare for the analysis of this method, we derive a singularity expansion of the exact solution of (1.1). In other words, we decompose the exact solution into two parts, one being a linear combination of singular functions  $t^{i+j\alpha}$  which reflect the singularity of the exact solution and the other being a smooth function. This subject has been well studied in [5]. We will make use of the results presented in [5] and construct further a form of expansion that is useful for the development of the HC method.

We organize this paper in five sections. In section 2, we derive the singularity expansion of the exact solution of (1.1). Section 3 is devoted to a study of hybrid interpolation operators which serve as a base for the development of the HC method. In section 4, we describe the HC method which combines the NPC method used near the singular point based on the singularity expansion obtained in section 2 and a GC method elsewhere. We prove the optimal order of global convergence of this method. Furthermore, we present a theoretical result which gives a comparison of the computational cost of the HC method and the GC method, and the length of the smallest subintervals used in both methods. Our theory shows that the HC method is better than the GC method. Finally in section 5, we provide a numerical example to demonstrate the effectiveness of the HC method. We compare the numerical performance of the HC method with that of the GC method. The numerical results confirm the theory presented in section 4.

**2. Singularity expansions.** In this section we establish a preliminary result on the singularity decomposition for the solution of (1.1). Singularity of the solution of (1.1) when the kernel  $M$  is zero has been systematically studied in [5]. In the next theorem, we make use of the results in [5] and derive the singularity expansion crucial for the development of the HC method for the general case when  $M \neq 0$ .

**THEOREM 2.1.** *Let  $r$  be a nonnegative integer. Suppose that  $K, M \in C^r(I \times I)$  and  $f$  has the form*

$$(2.1) \quad f(t) = \sum_{j+i\alpha < m} f_{ij} t^{j+i\alpha} + f_m(t), \quad t \in I,$$

where  $f_{ij}$  are constants and  $f_m \in C^m[0, 1]$  for some fixed integer  $m$  with  $0 \leq m \leq r$ . Let  $y$  denote the solution of (1.1). Then there exist constants  $c_{ij}$  such that

$$(2.2) \quad y(t) = \sum_{j+i\alpha < m} c_{ij} t^{j+i\alpha} + v_m(t), \quad t \in I,$$

where  $v_m \in C^m(I)$ .

*Proof.* When  $M \equiv 0$  formula (2.2) follows from Theorem 1.3.15 of [5] with some modification. The modification is necessary to treat the series in the expansion appearing in Theorem 1.3.15 of [5] so that we have form (2.2).

The general case where the kernel  $M$  is not zero will be proved by induction on  $m$ . The case when  $m = 0$  is obvious. We assume that the theorem holds for  $m = k$  and proceed to the case  $m = k + 1$ . By the induction hypothesis, the solution  $y$  of (1.1) has a representation

$$(2.3) \quad y(t) = \sum_{j+i\alpha < k} c_{ij} t^{j+i\alpha} + v_k(t), \quad t \in I,$$

where  $v_k \in C^k(I)$ . For  $t \in I$ , we let

$$x(t) := \int_0^t M(t, s)y(s)ds.$$

Substituting (2.3) into the expression of function  $x$  gives

$$x(t) = \int_0^t M(t, s) \left[ \sum_{j+i\alpha < k} c_{ij} s^{j+i\alpha} + v_k(s) \right] ds, \quad t \in I.$$

To simplify the expression of  $x$ , we denote

$$w_{k+1}(t) = \int_0^t M(t, s)v_k(s)ds \quad \text{and} \quad m_{ij}(t) = \int_0^1 M(t, st)s^{j+i\alpha}ds.$$

A simplification with a change of variables leads to the following formula:

$$x(t) = \sum_{j+i\alpha < k+1} c_{i,j-1} m_{i,j-1}(t) t^{j+i\alpha} + w_{k+1}(t), \quad t \in I.$$

It is easily seen that  $w_{k+1}, m_{ij} \in C^{k+1}(I)$ . Applying the Taylor theorem to the functions  $m_{ij}$ , we obtain that

$$x(t) = \sum_{j+i\alpha < k+1} d_{ij} t^{j+i\alpha} + u_{k+1}(t), \quad t \in I,$$

where  $d_{ij}$  are constants and  $u_{k+1}$  is a function in  $C^{k+1}(I)$ . Let  $\tilde{f} := f - x$  and rewrite (1.1) as

$$y(t) + \int_0^t (t - s)^{\alpha-1} K(t, s)y(s)ds = \tilde{f}(t), \quad t \in I.$$

Note that  $\tilde{f}$  has the form (2.1) with  $m = k + 1$ . By the first part of this proof, we conclude the result of the theorem for the case  $m = k + 1$ , which advances the induction hypothesis and completes the proof.  $\square$

**3. Nonpolynomial interpolation operators.** Motivated by the singularity expansion of the solution of (1.1), in the next section we will develop the HC method for solving (1.1). To prepare for this development, we define a hybrid interpolation operator and study the bound of this operator.

We first define a nonpolynomial finite dimensional subspaces of  $C(I)$ . As usual, we denote by  $N_0$  the set of nonnegative integers. For  $0 < \alpha < 1$  and a positive integer  $r$  we introduce an index set by setting

$$W_{\alpha,r} := \{i + j\alpha : i, j \in N_0, i + j\alpha < r\}.$$

Let  $\ell$  denote the cardinality of the set  $W_{\alpha,r}$ . Clearly,  $W_{\alpha,r}$  contains the first  $r$  non-negative integers  $i = 0, 1, \dots, r - 1$ . For notational convenience, we write

$$W_{\alpha,r} = \{\nu_j : j = 0, 1, \dots, \ell - 1\}$$

with the convention that  $\nu_j = j$  for  $j = 0, 1, \dots, r - 1$ . Associated with this index set, we define a finite dimensional space  $V_r$  of nonpolynomial functions by

$$V_r := \text{span}\{t^{\nu_j} : j = 0, 1, \dots, \ell - 1\}.$$

We remark that Theorem 2.1 ensures the solution  $y$  of (1.1) has the decomposition

$$y = u + v, \quad u \in C^r(I), \quad \text{and} \quad v \in V_r.$$

Also, we denote by  $P_r$  the space of polynomials of degree  $\leq r - 1$ . Because the set  $W_{\alpha,r}$  contains the integers  $i = 0, 1, \dots, r - 1$ , it is clear that  $P_r \subset V_r$ . In addition, space  $V_r$  contains nonpolynomial functions  $t^{\nu_j}$ ,  $j = r, r + 1, \dots, \ell - 1$ , that reflect the singularity of the derivative of the solution of (1.1).

We next describe a finite dimensional space whose elements are piecewise in  $V_r$ . For a given positive integer  $N$ , we divide the interval  $I$  into  $N$  subintervals, that is,  $0 = t_0 < t_1 < \dots < t_N = 1$ . For a subinterval  $J$  of  $I$  and a function  $f \in C(I)$ , we use  $f|_J$  for the restriction of  $f$  on  $J$  and, moreover, for  $V \subseteq C(I)$  we let

$$V|_J := \{v|_J : v \in V\}.$$

Let  $h_i = t_i - t_{i-1}$  and  $h = \max_{1 \leq i \leq N} h_i$ . We define a space of functions piecewise in  $V$  by

$$(V)_h := \{v : v|_{[t_{i-1}, t_i]} \in V|_{[t_{i-1}, t_i]}, \quad i = 1, 2, \dots, N\},$$

and, in particular, we let  $V_{r,h} := (V_r)_h$  and  $S_{r,h} := (P_r)_h$ . Clearly, we have that  $V_r \subseteq V_{r,h}$  and  $S_{r,h} \subseteq V_{r,h}$ .

We now define interpolation operators  $P_{h,1}$  from  $C(I)$  to  $S_{r,h}$  and  $P_{h,2}$  from  $C(I)$  to  $V_{r,h}$ , respectively. To this end, we choose  $\ell$  points  $\tau_j$  in  $I$  such that  $0 < \tau_1 < \tau_2 < \dots < \tau_\ell < 1$ . The interpolation points on interval  $[t_{k-1}, t_k]$  are obtained by setting  $t_{k,j} := t_{k-1} + \tau_j h_k$ ,  $k = 1, 2, \dots, N$ ,  $j = 1, 2, \dots, \ell$ . The interpolation operators  $P_{h,1} : C(I) \rightarrow S_{r,h}$  and  $P_{h,2} : C(I) \rightarrow V_{r,h}$  are defined as follows. For  $f \in C(I)$

$$(P_{h,1}f)(t_{ij}) = f(t_{ij}), \quad j = 1, 2, \dots, r, \quad i = 1, 2, \dots, N,$$

and

$$(P_{h,2}f)(t_{ij}) = f(t_{ij}), \quad j = 1, 2, \dots, \ell, \quad i = 1, 2, \dots, N.$$

Note that for the definition of  $P_{h,1}$  we use only the first  $r$  points  $\tau_j, j = 1, 2, \dots, r$ . It is known (see [2, 5]) that the operators  $P_{h,i}$  for  $i = 1, 2$  are uniquely defined.

For a function  $u$  that is continuous on  $[t_{i-1}, t_i], i = 1, 2, \dots, N$ , with possible discontinuities at  $t_i$ , define its maximum norm on  $[t_m, t_n]$  with  $0 \leq m < n \leq N$  by

$$\|u\|_{[t_m, t_n]} = \max_{m \leq i \leq n} \max_{t_{i-1} \leq t \leq t_i} |u(t)|.$$

We will simply use  $\|u\|$  when  $[t_m, t_n] = I$ .

Next we show that the norm of the restriction of  $P_{h,2}f$  on  $[t_0, t_1]$  is bounded by a constant independent of the choice of  $t_1$ . For this purpose, we define, for  $i = 1, 2, \dots, \ell$ , the Lagrange functions

$$L_{1i}|_{[t_0, t_1]} \in V_r|_{[t_0, t_1]}$$

and

$$L_{1i}(t) = 0, \quad t \in [t_1, 1]$$

such that

$$L_{1i}(t_{1j}) = \delta_{ij}, \quad j = 1, 2, \dots, \ell.$$

It is easily verified that

$$(3.1) \quad (P_{h,2}f)(t) = \sum_{j=1}^{\ell} f(t_{1j})L_{1j}(t), \quad t \in [t_0, t_1].$$

LEMMA 3.1. *There exists a positive constant  $c$  such that for all  $t_1 \in (0, 1)$  and  $i = 1, 2, \dots, \ell$*

$$\|L_{1i}\| = \|L_{1i}\|_{[t_0, t_1]} \leq c.$$

*Proof.* For  $i = 1, 2, \dots, \ell$ , we write

$$L_{1i}(t) = \sum_{p=1}^{\ell} a_{ip}t^{\nu_p}, \quad t \in [t_0, t_1].$$

For  $t \in [t_0, t_1]$ , there exists a  $\tau \in I$  such that  $t = t_0 + h_1\tau$ . For  $i = 1, 2, \dots, \ell$  and  $p = 1, 2, \dots, \ell$  we set  $b_{ip} := h_1^{\nu_p} a_{ip}$ . Using these notations, we have that

$$(3.2) \quad L_{1i}(t) = \sum_{p=1}^{\ell} b_{ip}\tau^{\nu_p}, \quad t \in [t_0, t_1].$$

By the definition of  $L_{1i}$  we observe that

$$(3.3) \quad \sum_{p=1}^{\ell} b_{ip}\tau_j^{\nu_p} = \delta_{ij}, \quad j = 1, 2, \dots, \ell.$$

We introduce an  $\ell \times \ell$  matrix  $\mathbf{D}$  by setting  $\mathbf{D} := [d_{jp} : j, p = 1, 2, \dots, \ell]$ , where  $d_{jp} := \tau_j^{\nu_p}$  and a vector  $\mathbf{b}_i$  by  $\mathbf{b}_i = [b_{ip} : p = 1, 2, \dots, \ell]^T$ . It is easy to verify that the matrix  $\mathbf{D}$  is invertible and since all entries  $d_{jp}$  are independent of the choice of  $t_1$

we conclude that  $\|\mathbf{D}^{-1}\|_\infty$  is bounded by a constant independent of the choice of  $t_1$  where the norm used here is the matrix norm induced from the vector norm  $\|\cdot\|_\infty$ . Thus, it follows from (3.3) that

$$\|\mathbf{b}_i\|_\infty \leq \|\mathbf{D}^{-1}\|_\infty.$$

Hence from (3.2) we confirm the result of this lemma with the constant

$$c := \max_{0 \leq \tau \leq 1} \left( \sum_{p=1}^{\ell} \tau^{\nu_p} \right) \|\mathbf{D}^{-1}\|_\infty. \quad \square$$

The following lemma, which provides a bound of the norm of  $P_{h,2}f$  on  $[t_0, t_1]$ , is a direct consequence of Lemma 3.1.

LEMMA 3.2. *There exists a positive constant  $c$  such that for all  $t_1 \in (0, 1)$  and for any  $f \in C(I)$*

$$\|P_{h,2}f\|_{[t_0, t_1]} \leq c\|f\|_{[t_0, t_1]}.$$

The next proposition presents order of convergence for the interpolation  $P_{h,2}f$  to a function  $f$  having form (2.1).

PROPOSITION 3.3. *There exists a positive constant  $c$  such that for all  $t_1 \in (0, 1)$  and for  $f = u + v$ , where  $u \in C^r(I)$  and  $v \in V_r$ ,*

$$\|f - P_{h,2}f\|_{[t_0, t_1]} \leq ch_1^r \|u^{(r)}\|_{[t_0, t_1]}.$$

*Proof.* For all functions  $f$  having the form  $f = u + v$ , where  $u \in C^r(I)$  and  $v \in V_r$ , recalling that  $P_{h,2}v = v$  for  $v \in V_r$  we have that

$$f - P_{h,2}f = u + v - P_{h,2}(u + v) = u - P_{h,2}u.$$

Noting that

$$P_{h,2}P_{h,1} = P_{h,1}$$

we conclude that

$$f - P_{h,2}f = (I - P_{h,1})u + P_{h,2}P_{h,1}u - P_{h,2}u = (I - P_{h,2})(I - P_{h,1})u.$$

It follows from Lemma 3.2 that there exists a positive constant  $c$  such that for all  $t_1 \in (0, 1)$  and all such  $f$

$$\|f - P_{h,2}f\|_{[t_0, t_1]} \leq c\|(I - P_{h,1})u\|_{[t_0, t_1]} \leq ch_1^r \|u^{(r)}\|_{[t_0, t_1]},$$

where the last inequality follows from a standard error estimate for polynomial interpolations.  $\square$

We next define a *hybrid* interpolation which has a *global* convergence. To this end, we describe a graded partition of  $I$  in terms of parameters  $\alpha$  and  $r$ . Specifically, for  $q := \frac{r}{\alpha}$  we let  $i_0$  be an integer such that

$$\left[ \left( \frac{N}{i_0} \right)^q \right] = N,$$



where  $[a]$  denotes the largest integer less than or equal to  $a$ , and set  $N' := N - i_0 + 1$ . The partition on  $I$  is given by

$$(3.4) \quad t_0 = 0, \quad t_i = \left( \frac{i_0 + i - 1}{N} \right)^q, \quad i = 1, 2, \dots, N'.$$

Note that  $t_{N'} = 1$  and the integer  $i_0$  satisfies the condition that

$$(3.5) \quad N^{1-1/q} \leq i_0 \leq N(N - 1)^{-1/q}.$$

We remark that as far as stability is concerned this partition is better than the partition (1.3) used in a standard GC method. This point will be made clearer in the next section.

Associated with the graded partition (3.4), we define the *hybrid interpolation* operator  $Q_h$  by

$$(3.6) \quad (Q_h f)|_{[0,t_1]} = (P_{h,2} f)|_{[0,t_1]} \quad \text{and} \quad (Q_h f)|_{[t_1,1]} = (P_{h,1} f)|_{[t_1,1]}.$$

That is, on the first subinterval we use singularity preserving (nonpolynomial) interpolation and on the rest of intervals we use the standard piecewise polynomial interpolation. The operator  $Q_h$  will be used in the next section for the development of a *hybrid collocation* method. To prepare for this development, we present an expression of projection  $Q_h$  in terms of the Lagrange basis functions. To this end, we define the Lagrange polynomial basis  $L_i \in P_r$ ,  $i = 1, 2, \dots, r$ , such that

$$L_i(\tau_j) = \delta_{ij}, \quad j = 1, 2, \dots, r,$$

and for  $k = 2, 3, \dots, N'$  we define the Lagrange piecewise polynomial basis functions by setting

$$L_{ki}(t) = \begin{cases} L_i \left( \frac{t-t_{k-1}}{h_k} \right), & t \in [t_{k-1}, t_k], \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for all  $k = 2, 3, \dots, N'$  there hold the relations that

$$(3.7) \quad \|L_{ki}\| = \|L_i\|, \quad i = 1, 2, \dots, r,$$

and we have that

$$(P_{h,1} f)(t) = \sum_{k=2}^{N'} \sum_{j=1}^r f(t_{kj}) L_{kj}(t), \quad t \in [t_1, 1].$$

To present projection  $Q_h$ , we introduce a notation

$$(3.8) \quad r_k := \begin{cases} \ell, & k = 1, \\ r, & k = 2, 3, \dots, N'. \end{cases}$$

Consequently, we have that

$$(3.9) \quad (Q_h f)(t) = \sum_{k=1}^{N'} \sum_{j=1}^{r_k} f(t_{kj}) L_{kj}(t), \quad t \in I.$$

In the next proposition, we establish a *global* convergence result for the interpolation projection  $Q_h$ .

**PROPOSITION 3.4.** *Let  $Q_h$  be the hybrid interpolation operator defined in (3.6) associated with the partition (3.4). Suppose that  $f$  has a decomposition  $f = u + v$ , where  $u \in C^r(I)$  and  $v \in V_r$ . Then there exists a positive constant  $c$  independent of  $N$  such that for all such functions  $f$*

$$\|f - Q_h f\| \leq cN^{-r}.$$

*Proof.* Note that

$$\|f - Q_h f\| = \max \{ \|f - Q_h f\|_{[0,t_1]}, \|f - Q_h f\|_{[t_1,1]} \}.$$

Employing Proposition 3.3 we conclude that there exists a positive constant  $c$  such that

$$\|f - Q_h f\|_{[0,t_1]} = \|f - P_{h,2} f\|_{[0,t_1]} \leq ch_1^r \|u^{(r)}\|_{[0,t_1]}.$$

Noting that by the definition of partition (3.4),

$$h_1 = t_1 = \left(\frac{i_0}{N}\right)^q \leq \left(\frac{N(N-1)^{-1/q}}{N}\right)^q = \frac{1}{N-1} \leq \frac{2}{N},$$

we obtain that

$$\|f - Q_h f\|_{[0,t_1]} \leq c \frac{1}{N^r}.$$

We next estimate the error  $\|f - Q_h f\|_{[t_1,1]}$  following a well-known argument by Rice [13]. Since  $f = u + v$  with  $u \in C^r(I)$  and  $v \in V_r$ , we have that

$$|f^{(r)}(t)| \leq ct^{\alpha-r} \quad \text{for } t \in [t_1, 1],$$

and the function  $t^{\alpha-r}$  is decreasing in  $t$ . For  $i = 2, 3, \dots, N'$ , we find that

$$h_i = [(i + i_0 - 1)^q - (i + i_0 - 2)^q] N^{-q} = (i + i_0 - 2)^q \left\{ [1 + (i + i_0 - 2)^{-1}]^q - 1 \right\} N^{-q}.$$

By the mean value theorem, there exists  $\theta$  with  $0 < \theta < (i + i_0 - 2)^{-1}$  such that

$$(3.10) \quad h_i = q(i + i_0 - 2)^{q-1} (1 + \theta)^{q-1} N^{-q} \leq c(i + i_0 - 2)^{q-1} N^{-q}.$$

Thus, there exists a positive constant  $c$  such that for  $i = 2, 3, \dots, N'$

$$\begin{aligned} \|f - Q_h f\|_{[t_{i-1}, t_i]} &= \|f - P_{h,2} f\|_{[t_{i-1}, t_i]} \leq ch_i^r \|f^{(r)}\|_{[t_{i-1}, t_i]} \\ &\leq c(i + i_0 - 2)^{(q-1)r} N^{-qr} \left(\frac{i + i_0 - 2}{N}\right)^{q(\alpha-r)} = c \frac{1}{N^r}, \end{aligned}$$

which completes the proof of this proposition.  $\square$

**4. A hybrid collocation method.** In this section, we use the hybrid interpolation operator  $Q_h$  introduced in the last section to develop a hybrid collocation method for solving (1.1). We prove that this method has an optimal order of global convergence. Notice that the singularity in the derivative of the exact solution of (1.1) occurs only at the left end point of the interval  $I$ . This fact suggests that we use a

singularity preserving collocation method near the left end point and use a standard piecewise polynomial collocation method with a graded partition elsewhere.

We now describe the hybrid collocation method for (1.1). We seek  $y_h$  such that

$$y_h|_{[0,t_1]} \in V_{r,h}|_{[0,t_1]}, \quad y_h|_{[t_1,1]} \in S_{r,h}|_{[t_1,1]},$$

and

$$(4.1) \quad y_h - Q_h T_\alpha y_h = Q_h f,$$

where  $Q_h$  is the hybrid interpolation operator defined by (3.6) associated with the graded partition (3.4).

To analyze the order of convergence for the hybrid collocation method (4.1), we define an integral operator  $T_{\alpha,1}$  by

$$(4.2) \quad (T_{\alpha,1}y)(t) := (T_\alpha y)(t) \quad \text{for } t \in [0, t_1]$$

and

$$(T_{\alpha,1}y)(t) := \int_{t_1}^t G_\alpha(t, s)y(s)ds \quad \text{for } t \in [t_1, 1].$$

The study of the error  $\|y - y_h\|$  demands a bound on the errors

$$e_{kj} := y(t_{kj}) - y_h(t_{kj}), \quad j = 1, 2, \dots, r_k,$$

where  $r_k$  is defined by (3.8). To this end, we introduce vectors

$$\mathbf{e}_k := [e_{kj} : j = 1, 2, \dots, r_k]^T \quad \text{for } k = 1, 2, \dots, N'.$$

We also need vectors

$$\epsilon_k := [\epsilon_{kj} : j = 1, 2, \dots, r_k]^T \quad \text{for } k = 1, 2, \dots, N',$$

where

$$(4.3) \quad \epsilon_{kj} := (T_{\alpha,1}(y - Q_h y))(t_{kj}) + ((T_\alpha - T_{\alpha,1})(y - y_h))(t_{kj}).$$

Note that when  $k = 1$ , it becomes that

$$\epsilon_{1j} = (T_\alpha(y - Q_h y))(t_{1j}).$$

We will bound the vectors  $\mathbf{e}_k$  by  $\epsilon_k$ .

Next, we derive a linear system that gives a recursive formula for the vector  $\mathbf{e}_k$ . Toward this goal, for  $k = 1, 2, \dots, N'$  and  $j = 1, 2, \dots, r_k$  we evaluate (1.1) and (4.1) at  $t_{kj}$  to obtain that

$$(4.4) \quad y(t_{kj}) - (T_\alpha y)(t_{kj}) = f(t_{kj})$$

and

$$(4.5) \quad y_h(t_{kj}) - (T_\alpha y_h)(t_{kj}) = f(t_{kj}),$$

respectively. Subtracting (4.5) from (4.4) yields

$$(4.6) \quad e_{kj} = (T_{\alpha,1}(y - y_h))(t_{kj}) + ((T_\alpha - T_{\alpha,1})(y - y_h))(t_{kj}).$$

Noticing that  $Q_h$  is a projection and  $y_h = Q_h y_h$ , we have that

$$(4.7) \quad y - y_h = y - Q_h y + Q_h(y - y_h).$$

Substituting (4.7) into the first term in the right-hand side of (4.6) and recalling the definition of  $\epsilon_{kj}$ , we obtain that

$$(4.8) \quad e_{kj} = (T_{\alpha,1} Q_h(y - y_h))(t_{kj}) + \epsilon_{kj} \quad \text{for } k = 1, 2, \dots, N', \quad j = 1, 2, \dots, r_k.$$

We are required to study the first term in the right-hand side of (4.8). In (3.9), we replace  $f$  by  $y - y_h$  and conclude that

$$(Q_h(y - y_h))(t) = \sum_{i=1}^{N'} \sum_{p=1}^{r_i} e_{ip} L_{ip}(t), \quad t \in I.$$

Applying the operator  $T_{\alpha,1}$  to both sides of this equation with evaluating at  $t = t_{kj}$  yields

$$(T_{\alpha,1} Q_h(y - y_h))(t_{kj}) = \sum_{i=1}^{N'} \sum_{p=1}^{r_i} e_{ip} (T_{\alpha,1} L_{ip})(t_{kj}).$$

We next make use of the property of the Lagrange basis functions  $L_{ip}$  to simplify the right-hand side of the equation above. For notational convenience, we define

$$a_{jp}^k := \int_{t_{k-1}}^{t_{kj}} G_\alpha(t_{kj}, s) L_{kp}(s) ds$$

and

$$d_{jp}^{ki} := \int_{t_{i-1}}^{t_i} G_\alpha(t_{kj}, s) L_{ip}(s) ds.$$

Noting that  $L_{ip}$  vanishes outside the interval  $[t_{i-1}, t_i]$ , an elementary computation leads to the formula that

$$(T_{\alpha,1} Q_h(y - y_h))(t_{kj}) = \sum_{p=1}^{r_k} a_{jp}^k e_{kp} + \sum_{i=2}^{k-1} \sum_{p=1}^r d_{jp}^{ki} e_{ip},$$

where we have used the relation that  $r_i = r$  for  $i = 2, 3, \dots, k-1$ . Substituting this equation into the right-hand side of (4.8) yields

$$(4.9) \quad e_{kj} = \sum_{p=1}^{r_k} a_{jp}^k e_{kp} + \sum_{i=2}^{k-1} \sum_{p=1}^r d_{jp}^{ki} e_{ip} + \epsilon_{kj}, \quad j = 1, 2, \dots, r_k.$$

By introducing an  $r_k \times r_k$  matrix

$$(4.10) \quad \mathbf{A}_k := [a_{jp}^k : j, p = 1, 2, \dots, r_k]$$

and  $r \times r$  matrices

$$(4.11) \quad \mathbf{D}_{ki} := [d_{jp}^{ki} : j, p = 1, 2, \dots, r], \quad i = 2, 3, \dots, k-1,$$

we write (4.9) in matrix form as

$$(4.12) \quad \mathbf{e}_k = \mathbf{A}_k \mathbf{e}_k + \sum_{i=2}^{k-1} \mathbf{D}_{ki} \mathbf{e}_i + \epsilon_k, \quad k = 1, 2, \dots, N'.$$

The matrices  $\mathbf{A}_k$  and  $\mathbf{D}_{ki}$  are all dependent on the mesh sizes  $h_i$ , and we next study such a dependence. Recalling the transformations  $s = t_{k-1} + h_k \tau$  when  $s \in [t_{k-1}, t_k]$  and  $\tau \in [0, 1]$  and  $t_{kj} = t_{k-1} + h_k \tau_j$  and using the notations

$$\tilde{K}(t, \tau) := K(t, t_{k-1} + h_k \tau), \quad \tilde{M}(t, \tau) := M(t, t_{k-1} + h_k \tau), \quad \tilde{L}_{kp}(\tau) := L_{kp}(t_{k-1} + h_k \tau),$$

$$\tilde{a}_{jp}^k := \int_0^{\tau_j} [(\tau_j - \tau)^{\alpha-1} \tilde{K}(t_{kj}, \tau) + h_k^{1-\alpha} \tilde{M}(t_{kj}, \tau)] \tilde{L}_{kp}(\tau) d\tau,$$

and

$$\tilde{d}_{jp}^{ki} := \int_0^1 \left[ \left( \frac{t_{k-1} - t_{i-1} + h_k \tau_j - \tau}{h_i} \right)^{\alpha-1} \tilde{K}(t_{kj}, \tau) + h_i^{1-\alpha} \tilde{M}(t_{kj}, \tau) \right] \tilde{L}_{ip}(\tau) d\tau,$$

by changes of variables we have that

$$a_{jp}^k = h_k^\alpha \tilde{a}_{jp}^k \quad \text{and} \quad d_{jp}^{ki} = h_i^\alpha \tilde{d}_{jp}^{ki}.$$

By introducing new matrices

$$\tilde{\mathbf{A}}_k := [\tilde{a}_{jp}^k : j, p = 1, 2, \dots, r_k] \quad \text{and} \quad \tilde{\mathbf{D}}_{ki} := [\tilde{d}_{jp}^{ki} : j, p = 1, 2, \dots, r], \quad i = 2, 3, \dots, k-1,$$

we observe that

$$\tilde{\mathbf{A}}_k = h_k^{-\alpha} \mathbf{A}_k \quad \text{and} \quad \tilde{\mathbf{D}}_{ki} = h_k^{-\alpha} \mathbf{D}_{ki}.$$

Hence, (4.12) becomes

$$(4.13) \quad \mathbf{e}_k = h_k^\alpha \tilde{\mathbf{A}}_k \mathbf{e}_k + \sum_{i=2}^{k-1} h_i^\alpha \tilde{\mathbf{D}}_{ki} \mathbf{e}_i + \epsilon_k, \quad k = 1, 2, \dots, N'.$$

Since both  $\tilde{K}$  and  $\tilde{M}$  are continuous functions on  $I \times I$ , by using Lemma 3.1 and (3.7) we observe that there exists a positive constant  $c_1$  such that for  $k = 1, 2, \dots, N'$  and  $j, p = 1, 2, \dots, r_k$

$$(4.14) \quad |\tilde{a}_{jp}^k| \leq c_1 \int_0^{\tau_j} [(\tau_j - \tau)^{\alpha-1} + 1] d\tau \leq c_1 \left( 1 + \frac{1}{\alpha} \right)$$

and

$$(4.15) \quad |\tilde{d}_{jp}^{ki}| \leq c_1 \int_0^1 \left( \frac{t_{k-1} - t_{i-1} + h_k \tau_j - \tau}{h_i} \right)^{\alpha-1} d\tau, \quad i = 2, 3, \dots, k-1.$$

Estimate (4.14) implies that there exists a positive constant  $c_2 := c_1 \ell \left( 1 + \frac{1}{\alpha} \right)$  for all  $k = 1, 2, \dots, N'$ ,

$$\|\tilde{\mathbf{A}}_k\|_\infty \leq c_2.$$

Let  $h := \max\{h_i : 1 \leq i \leq N'\}$  and choose  $h < c_2^{-1/\alpha}$ . It follows that for such an  $h$  the matrix  $I - h_k^\alpha \tilde{\mathbf{A}}_k$  is invertible and there exists a positive constant  $c$  such that

$$(4.16) \quad \|(I - h_k^\alpha \tilde{\mathbf{A}}_k)^{-1}\| \leq c.$$

Thus, from (4.13) we conclude that

$$(4.17) \quad \mathbf{e}_k = (I - h_k^\alpha \tilde{\mathbf{A}}_k)^{-1} \left[ \sum_{i=2}^{k-1} h_i^\alpha \tilde{\mathbf{D}}_{ki} \mathbf{e}_i + \epsilon_k \right].$$

We next use estimate (4.15) to study the bound of the entries of matrices  $\tilde{\mathbf{D}}_{ki}$ . Noting that  $h_i < h_{i+1} < \dots < h_{k-1}$ , we have that

$$\frac{t_{k-1} - t_{i-1}}{h_i} = \frac{h_i + \dots + h_{k-1}}{h_i} \geq k - i$$

and conclude that

$$(4.18) \quad |\tilde{d}_{jp}^{ki}| \leq c_1 \int_0^1 (k - i - \tau)^{\alpha-1} d\tau.$$

It can be verified from a direct computation that there exists a positive constant  $c$  such that for all  $i = 2, 3, \dots, k-1$

$$\int_0^1 (k - i - \tau)^{\alpha-1} d\tau \leq c(k - i)^{\alpha-1}.$$

Using this estimate in inequality (4.18) yields for  $i = 2, 3, \dots, k-1$  and  $j, p = 1, 2, \dots, r_k$  that

$$|\tilde{d}_{jp}^{ki}| \leq c(k - i)^{\alpha-1}$$

and thus

$$(4.19) \quad \|\tilde{\mathbf{D}}_{ki}\|_\infty \leq c(k - i)^{\alpha-1}.$$

Combining estimates (4.16) and (4.19) with (4.17) gives

$$\|\mathbf{e}_k\|_\infty \leq c \sum_{i=2}^{k-1} h_i^\alpha (k - i)^{\alpha-1} \|\mathbf{e}_i\|_\infty + c \|\epsilon_k\|_\infty.$$

Recalling from (3.10) that there exists a positive constant  $c$  such that

$$h_i \leq c(i + i_0 - 2)^{q-1} N^{-q},$$

since  $i + i_0 - 2 \leq 2N$  we conclude that there exists a positive constant  $c$  such that

$$h_i \leq cN^{-1}.$$

Therefore, we have that

$$(4.20) \quad \|\mathbf{e}_k\|_\infty \leq c \left( \frac{1}{N} \right)^\alpha \sum_{i=2}^{k-1} (k - i)^{\alpha-1} \|\mathbf{e}_i\|_\infty + c \|\epsilon_k\|_\infty.$$

We next use inequality (4.20) to obtain the error estimate of the hybrid collocation method (4.1). For this purpose, we recall a discrete Gronwall-type inequality (cf. [5]).

LEMMA 4.1. *Let  $0 < \alpha < 1$  and  $\{z_i : i = 1, 2, \dots, n\}$  be a sequence of positive numbers and  $n \leq N$ . Let  $\rho$  and  $\beta$  be two positive numbers such that for all  $k = 1, 2, \dots, n$*

$$z_k \leq \left(\frac{1}{N}\right)^\alpha \beta \sum_{i=1}^{k-1} (k-i)^{\alpha-1} z_i + \rho.$$

*Then there exists a positive constant  $c$  depending only on  $\alpha$  and  $\beta$  such that  $k = 1, 2, \dots, n$ ,*

$$z_k \leq c\rho.$$

We are now ready to prove the main result of this paper, which gives an optimal order of global convergence for the hybrid collocation method.

THEOREM 4.2. *Let  $y$  be the exact solution of (1.1), let  $N$  be a positive integer, and let  $Q_h$  be the hybrid interpolation operator defined by (3.6) associated with the graded partition (3.4). Suppose that the forcing function  $f$  in (1.1) has the form (2.1). Then, for sufficiently large  $N$ , (4.1) has a unique solution  $y_h$  and there exists a positive constant  $c$  independent of  $N$  such that*

$$\|y - y_h\| \leq cN^{-r}.$$

*Proof.* It follows from Theorem 2.1 that the solution of (1.1) has the form  $y = u + v$ , where  $u \in C^r(I)$  and  $v \in V_r$ . This allows us to use Proposition 3.4 to prove the result. We first estimate the error on  $[t_0, t_1]$ . From (4.3) there exists a constant  $c$  such that

$$\|\epsilon_1\|_\infty \leq \|T_\alpha(y - Q_h y)\|_{[t_0, t_1]} \leq c\|y - Q_h y\|_{[t_0, t_1]}.$$

It follows from (4.20) and Proposition 3.4 that

$$\|\mathbf{e}_1\|_\infty \leq c\|y - Q_h y\|_{[t_0, t_1]} \leq cN^{-r}.$$

Using Lemma 3.1 and the above estimate we obtain that

$$\|Q_h(y - y_h)\|_{[t_0, t_1]} \leq c\|\mathbf{e}_1\|_\infty \leq cN^{-r},$$

which with Proposition 3.4 gives that

$$(4.21) \quad \|y - y_h\|_{[t_0, t_1]} \leq \|Q_h(y - y_h)\|_{[t_0, t_1]} + \|Q_h y - y\|_{[t_0, t_1]} \leq cN^{-r}.$$

Next we estimate the error on  $[t_1, 1]$ . Using (4.3) we have that for  $k = 2, 3, \dots, N'$

$$\|\epsilon_k\|_\infty \leq \|T_{\alpha,1}(y - Q_h y)\|_{[t_1, 1]} + \|(T_\alpha - T_{\alpha,1})(y - y_h)\|_{[t_1, 1]}.$$

Now, by using Proposition 3.4 and estimate (4.21) we conclude that there exists a positive constant  $c$  such that

$$\|\epsilon_k\|_\infty \leq c(\|y - Q_h y\|_{[t_1, 1]} + \|y - y_h\|_{[t_0, t_1]}) \leq cN^{-r}.$$

Combining the above estimate with (4.20) we obtain that

$$\|\mathbf{e}_k\|_\infty \leq cN^{-\alpha} \sum_{i=2}^{k-1} (k-i)^{\alpha-1} \|\mathbf{e}_i\|_\infty + cN^{-r}.$$

Using Lemma 4.1 with  $z_1 = 0$  and  $z_i = \|\mathbf{e}_i\|_\infty$  for  $i = 2, 3, \dots, N'$ , we conclude that there exists a positive constant  $c$  such that for all  $k = 2, 3, \dots, N'$

$$\|\mathbf{e}_k\|_\infty \leq cN^{-r}.$$

Now, by the uniform boundedness (3.7) of  $\|L_{ij}\|$  for  $i = 2, 3, \dots, N'$  there exists a positive constant  $c$  such that

$$\|Q_h(y - y_h)\|_{[t_1, 1]} \leq c \max\{\|\mathbf{e}_k\|_\infty : k = 2, 3, \dots, N'\} \leq cN^{-r}.$$

It follows that there exists a positive constant  $c$  such that

$$\|y - y_h\|_{[t_1, 1]} \leq \|Q_h(y - y_h)\| + \|Q_h y - y\| \leq cN^{-r},$$

which concludes the proof of the theorem.  $\square$

We may also use the compactness of operator  $T_\alpha$  and the uniform boundedness of  $Q_h$  to prove Theorem 4.2 (see, for example, [1]). In fact, such a proof is more concise. We choose the current proof, for it provides guidance for the construction of a numerical algorithm in our numerical experiments.

In the next proposition, we compare the graded collocation (GC) method with the hybrid collocation (HC) method. To this end, we let  $\mathcal{N}_{GC}$  and  $\mathcal{N}_{HC}$  denote the number of subintervals used in the GC method and the HC method, and we let  $\mathcal{L}_{GC}$  and  $\mathcal{L}_{HC}$  denote the length of the smallest subinterval used in the GC method and the HC method, respectively. We also consider the ratios of the largest subinterval over the smallest subinterval for the partitions that associate with the GC method and the HC method, which are denoted by  $\mathcal{R}_{GC}$  and  $\mathcal{R}_{HC}$ . Such a ratio is a good measure for the stability of the corresponding collocation method.

PROPOSITION 4.3. *There hold the estimates that*

$$\begin{aligned} \mathcal{N}_{GC} - \mathcal{N}_{HC} &\geq \frac{N}{\sqrt[q]{N}} - 1, \\ \mathcal{L}_{GC} &= \frac{1}{N^q}, \quad \mathcal{L}_{HC} \geq \frac{\sqrt[q]{N}}{N^2}, \end{aligned}$$

and

$$\mathcal{R}_{GC} \geq q(N-1)^{q-1}, \quad \mathcal{R}_{HC} \leq \frac{qN}{\sqrt[q]{N}}.$$

*Proof.* Since  $\mathcal{N}_{GC} = N$  and  $\mathcal{N}_{HC} = N - i_0 + 1 \leq N - N^{1-1/q} + 1$ , we have that

$$\mathcal{N}_{GC} - \mathcal{N}_{HC} \geq N - (N - N^{1-1/q} + 1) = \frac{N}{\sqrt[q]{N}} - 1.$$

The smallest interval used in the GC method is  $[0, N^{-q}]$  and thus  $\mathcal{L}_{GC} = \frac{1}{N^q}$ . For the HC method, it is easily verified that  $\mathcal{L}_{HC} = \min\{h_1, h_2\}$ , where  $h_1 = \left(\frac{i_0}{N}\right)^q$  and  $h_2 = \left(\frac{i_0+1}{N}\right)^q - \left(\frac{i_0}{N}\right)^q$ . Recalling that  $i_0$  satisfies that condition  $N^{1-\frac{1}{q}} \leq i_0 \leq N(N-1)^{-\frac{1}{q}}$ , we derive that

$$h_1 \geq \left(\frac{N^{1-\frac{1}{q}}}{N}\right)^q = \frac{1}{N} \geq \frac{\sqrt[q]{N}}{N^2}.$$



On the other hand, there exists a constant  $\theta$  with  $0 < \theta < 1/i_0$  such that

$$h_2 = qi_0^{q-1}(1 + \theta)^{q-1}N^{-q}.$$

Thus

$$h_2 \geq qi_0^{q-1}N^{-q} \geq qN^{\frac{1}{q}-2} \geq \frac{\sqrt[q]{N}}{N^2}.$$

This concludes the second estimate in this proposition. The third estimate can be similarly obtained.  $\square$

We remark that it follows from Proposition 4.3 that the HC method requires less computational cost than the GC method even though they have the same order of convergence. For example, when  $\alpha = \frac{1}{2}$ ,  $r = 3$ , and  $N = 100$ , the HC method uses 54 subintervals while the GC method uses 100 subintervals.

Another important point made in the last proposition is that the HC method is more stable than the GC method since the length of the smallest subinterval used in the HC method is larger than the length of the smallest subinterval used in the GC method. Notice that the length of the smallest interval used in the HC method is not as sensitive to  $r$  and  $\alpha$  as that in the GC method. For instance, when  $N = 1000$ ,  $\alpha = \frac{1}{2}$ , and  $r = 3$ , the length of the smallest interval used in the GC method is  $h_1 = 10^{-24}$  while the length of the smallest interval used the HC method is  $4.217 \times 10^{-5}$ . In addition, we see from the proposition that for the GC method  $\mathcal{R}_{GC}$  grows in the order  $\mathcal{O}(N^{q-1})$  while for the HC method  $\mathcal{R}_{HC}$  grows slower than  $\mathcal{O}(N)$ . When  $p$  is large, which is the case when  $\alpha$  is small or  $r$  is large,  $\mathcal{R}_{GC}$  is extremely large. This may cause serious instability problems. The result in the proposition shows that the HC method is much more stable than the GC method.

**5. Numerical experiments.** In this section, we report results of numerical experiments which confirm the theoretical analysis for the HC method presented in the last section and demonstrate the effectiveness of the method.

In (1.1) we choose  $K(s, t) = M(s, t) = 1$ ,  $\alpha = 1/2$  and choose  $f$  such that the equation has the exact solution

$$y(t) = \sqrt{t^2 + t} \cos t + \sin t, \quad t \in I.$$

Note that the first derivative of this solution has a singularity at  $t = 0$ .

The purpose of these numerical experiments is to compare the numerical performance of the HC method with the GC method. For both of the methods we use piecewise polynomials of degree 2, that is,  $r = 3$ , and, in addition, for the HC method we use

$$V_3 = \text{span}\{1, t, t^2, t^{\frac{1}{2}}, t^{\frac{3}{2}}, t^{\frac{5}{2}}\}$$

on the first interval.

Tables 5.1 and 5.2 are given to compare the numerical performance of the two methods, where ‘‘order of conv.’’ stands for the order of convergence. The weakly singular integrals that appear in these methods are computed by a numerical integration scheme presented in [12] specifically designed for weakly singular integrals of this type.

The HC method and the GC method have the same orders of convergence. The computed orders of convergence are consistent with the theoretical order, which is  $r =$

TABLE 5.1  
*Numerical performance of the HC method.*

$N$	$\ y - y_h\ $	Order of conv.	$\mathcal{N}_{HC}$	$\mathcal{L}_{HC}$
20	3.666e-4	-	8	7.5419e-02
40	4.680e-5	2.9698	19	2.7681e-02
60	1.408e-5	2.9616	30	1.9022e-02
80	5.983e-6	2.9759	42	1.3423e-02
100	3.069e-6	2.9916	54	1.0779e-02

TABLE 5.2  
*Numerical performance of the GC method.*

$N$	$\ y - y_h\ $	Order of conv.	$\mathcal{N}_{GC}$	$\mathcal{L}_{GC}$
20	1.993e-4	-	20	1.5625e-08
40	3.446e-5	2.9720	40	2.4414e-10
60	1.240e-5	2.9576	60	2.1433e-11
80	6.011e-6	2.9776	80	3.8147e-12
100	3.430e-6	2.9896	100	1.0000e-12

3 for both methods. In terms of convergence both methods give satisfactory numerical performance. However, the HC method uses much fewer subintervals. Therefore, it requires less computational cost than the GC method. Also, the length of the smallest subinterval used in the HC method is significantly larger than that in the GC method. When  $N$  is large and  $\alpha$  is small, for the GC method, the length of the first interval is extremely small, which may cause serious round-off errors. The HC method has a rather uniform partition, which avoids the problem of having small subintervals. In these two aspects, the HC method has performed better than the GC method.

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