Spring 2021

Algebra

WVU Mathematics Department

West Virginia University
DEPARTMENT OF MATHEMATICS

M.S. Advanced/Ph.D. Entrance exam in Algebra

April 2021

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	☐	☐	☐	☐	☐	☐	☐	☐	☐	
Pages										
Score										

PLEASE READ THE DIRECTIONS CAREFULLY:

This exam has three parts:
Part A: Group Theory, **Part B**: Field and Galois Theory, **Part C**: Ring and Module Theory.

** *SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B, AND C* **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded**.
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit.
Do not submit scratchworks and solutions that are not to be graded.
Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences using correct English grammar. *Answers, even correct, without justifications will not receive full credit*.
- **If you make use of a result which you do not prove, write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

Part A. Group Theory

Conventions. For a given group G,

- $Z(G)$ denotes the *center* of G.
- G is called simple if $G \neq \{e\}$ and the only normal subgroups of G are $\{e\}$ and G.

Questions.

(1) Let G be a group such that $|G| = 102 = 2 \cdot 3 \cdot 17$.
Assume there are elements $x, y \in Z(G)$ such that $|x| = 2$ and $|y| = 3$.
Prove that G is Abelian.

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(2) Let G be a finite Abelian group, and let p be a prime integer dividing the order $|G|$ of G.
Use induction on $|G|$ and prove that G has an element of order p.

(Note: In order to receive credit, you must give a proof by using induction on $|G|$. You are not allowed to use Cauchy's theorem, or Sylow's theorems for this problem as the aim of the problem is to establish Cauchy's theorem in the Abelian case. However, you can use, without proof, that each finite cyclic group has an element of order p if p divides its order.)

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(3) Let G be a group of order $90 = 2 \cdot 3^2 \cdot 5$. Prove that G is *not* simple.

(Hint: One way is to assume G is simple, and seek a contradiction. Show that there are two distinct Sylow 3-subgroups of G, say P and Q, such that $|P \cap Q| = 3$. Show $PQ \subseteq N_G(X)$, where $X = P \cap Q$ and $N_G(X) = \{g \in G : gXg^{-1} = X\}$ is the normalizer of X in G. Look for non-trivial, proper normal subgroups of G by considering $|X|$ and $[G : X]$.)

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Part B. Field and Galois Theory

Conventions.

- $\mathbb{Q}$ denotes the set of rational numbers.
- A *Galois* extension is a field extension that is finite, normal, and separable.

Questions.

(4) Consider the field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$, where $\alpha = \sqrt{1+\sqrt{3}}$.
Determine whether or not this field extension is *normal*.

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(5) Let R be a commutative ring, and let $\mathfrak{p}$ be a prime ideal of R which is *not* a maximal ideal of R.
Prove that the *field of fractions* $\mathrm{Frac}(R/\mathfrak{p})$ of the ring $R/\mathfrak{p}$ is an *infinite field*.

$\big($Hint: Consider the natural ring map $R/\mathfrak{p} \to \mathrm{Frac}(R/\mathfrak{p})$.$\big)$

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(6) Let K/F be a Galois extension and let L be an intermediate field of K/F, i.e., $F \subseteq L \subseteq K$.
Set $G = Gal(K/F)$ and $H = \mathrm{Gal}(K/L)$. Assume the following property holds:

If E is an intermediate field such that $F \subseteq E \subseteq K$ and $[L:F]$ divides $[E:F]$, it follows $L \subseteq E$.

Prove the following:

(i) If X is a subgroup of G such that $|X|$ divides $|H|$, then $X \subseteq H$.

(ii) Use part (i) and prove that L/F is a Galois extension.

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Part C. Ring and Module Theory

Conventions.

- R denotes a ring (not necessarily commutative) which has *multiplicative identity* 1 such that $1 \neq 0$. Moreover, all R-modules are assumed to be left modules.

Questions.

(7) Let R be a ring.
Assume the following condition holds: if $x \in R$ and $x^2 = 0$, then $x = 0$.
Prove R has no nonzero nilpotent element, i.e., if $a \in R$ and $a^r = 0$ for some $r \geq 1$, then $a = 0$.

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(8) Let R be a commutative ring and let $x \in R$. Assume the following conditions hold:

(i) $0 \to K \xrightarrow{f} N \xrightarrow{g} M$ is an exact sequence of R-modules, i.e., K, N and M are R-modules, f and g are R-module homomorphisms, f is injective, and $\text{im}(f) = \ker(g)$.

(ii) x is a non zero-divisor on M, i.e., if $m \in M$ and $xm = 0$, then $m = 0$.

Consider the map $\overline{f} : K/xK \to N/xN$ defined by $\overline{f}(k + xK) = f(k) + xN$ for each $k \in K$.
Prove that $\overline{f}$ is injective.

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(9) Let R be a ring, M be an R-module, and let $f : M \to M$ be a (left) R-module homomorphism.
Assume M is both Artinian and Noetherian as a (left) R-module.
Prove that there exists a positive integer n such that M is the direct sum of $\text{im}(f^n)$ and $\ker(f^n)$, i.e., $M = \text{im}(f^n) \bigoplus \ker(f^n)$, where f^n denotes the composition of f with itself n-times.

(Hint: consider the submodules $\text{im}(f^i)$ and $\ker(f^i)$ for $i = 1, 2, \ldots$)

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Write **big** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page #   of problem # | name: