Fall 2022

Algebra

WVU Mathematics Department

WestVirginiaUniversity
DEPARTMENT OF MATHEMATICS

M.S. Advanced/Ph.D. Entrance Exam in Algebra

September 2022

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
√	☐	☐	☐	☐	☐	☐	☐	☐	☐	
Pages										
Score										

PLEASE READ THE DIRECTIONS CAREFULLY:

This exam has *three* parts:
Part A: Group Theory, **Part B**: Field and Galois Theory, **Part C**: Ring and Module Theory.

** *SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B, AND C* **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded**.
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **only one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit. **Do not submit any scratchwork and solutions that are not to be graded**. Return your solutions with pages in correct order arranged according to the problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded if they cannot be read. Justify your arguments with complete sentences. *Answers, even correct, without justification will not receive (full) credit*.
- **If you make use of a result which you do not prove, then write separately the complete statement of the result you use underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

Part A. Group Theory

Questions.

(1) Let G be a finite p-group, that is, $|G| = p^n$ for some prime number p and positive integer n. Prove that G has a *normal* subgroup of order p.

(Hint: you may consider the center $\mathrm{Z}(G)$ of G.)

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(2) Let G be a finite group such that the following hold:
- (i) G is simple.
- (ii) The order of G is at least six, that is, $|G| \geq 6$.
- (iii) G *does not* contain any element of even order.

Prove that, if H is a subgroup of G, then the index $[G : H]$ of H in G is not 5.

(Hint: if there is such a subgroup H, then consider an action of G on the set of all left cosets of H.)

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(3) Let G be a group such that $|G| = 3 \cdot 17 \cdot 19$. Prove that G is *solvable*.

(Hint: first show that a group of order $3 \cdot 17$ is solvable.)

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Part B. Field and Galois Theory

Questions.

(4) Let $F \subsetneq K$ be a field extension and let $\beta \in K - F$.
Assume $K = F(\alpha)$ for some $\alpha \in K$.
Prove that α is *algebraic* over $F(\beta)$.

(Hint: $\beta = \dfrac{f(\alpha)}{g(\alpha)}$ for some polynomials $f(x) \in F[x]$ and $g(x) \in F[x]$ such that $g(\alpha) \neq 0$.)

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(5) Let E/F be a *separable* field extension.
Assume there is an integer n such that $[F(\gamma) : F] \leq n$ for each $\gamma \in E$.
Prove that E/F is a *simple* field extension, that is, $E = F(\alpha)$ for some $\alpha \in E$.

(Hint: recall that each separable field extension is algebraic by our definition, and each *finite* and *separable* field extension is *simple* by the Primitive Element Theorem. Observe there is an element $\alpha \in E$ such that $[F(\alpha) : F]$ is as large as possible.)

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(6) Determine all *intermediate* field(s) of the Galois extension $\mathbb{Q}(\omega)/\mathbb{Q}$, where ω is a *primitive fifth root of unity*. Give a generator for each of the intermediate field(s) you find.

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Part C. Ring and Module Theory

Convention.

• In this section, R denotes a commutative ring which has *multiplicative identity* 1 such that $1 \neq 0$. Moreover, all R-modules are assumed to be left R-modules.

Questions.

(7) Consider the following *commutative* diagram of R-modules and R-module homomorphisms with exact rows. In other words, we have:

(a) A, B, C, A', B', C' are R-modules, and $\psi, \Phi, \alpha, \beta, \gamma, \psi', \Phi'$ are R-module homomorphisms.
(b) $\beta\psi = \psi'\alpha$ and $\gamma\Phi = \Phi'\beta$.
(c) $\mathrm{im}(\psi) = \ker(\Phi)$ and $\mathrm{im}(\psi') = \ker(\Phi')$.

Here the operation between homomorphisms is the composition, that is, $\beta\psi$ means the composition of β and ψ. Also, im denotes the *image*, and ker denotes the *kernel* of the homomorphism.

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\Phi} & C \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ A' & \xrightarrow{\psi'} & B' & \xrightarrow{\Phi'} & C' \end{array}$$

Assume Φ, α and γ are surjective. Prove that β is surjective.

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(8) Assume R is local, that is, R has a *unique* maximal ideal.
Assume further $e \in R$ is an idempotent element, that is, $e^2 = e$.
Prove that $e = 0$ or $e = 1$.

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(9) Let M be an R-module and let N be a *finitely generated* R-module.
Let $f : M \to N$ be a (left) R-module homomorphism.
Assume $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ (the map f localized at $\mathfrak{p}$) is *surjective* for all prime ideals $\mathfrak{p}$ of R.
Prove that f is surjective.

(Hint: consider the cokernel of the map f.)

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Write **big** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page # of problem # | name: