Spring 2022

Algebra

WVU Mathematics Department

WestVirginiaUniversity
DEPARTMENT OF MATHEMATICS

M.S. Advanced/Ph.D. Entrance Exam in Algebra

April 2022

Part	A			B			C			Total Score
#	1	2	3	4	5	6	7	8	9	
✓	☐	☐	☐	☐	☐	☐	☐	☐	☐	
Pages										
Score										

PLEASE READ THE DIRECTIONS CAREFULLY:

This exam has *three* parts:
Part A: Group Theory, **Part B**: Field and Galois Theory, **Part C**: Ring and Module Theory.

** *SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B, AND C* **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7, and 8 will be graded**.
- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.
- Write the solution on **one side** of the paper and stay within the borders. Anything written outside the borders will not be taken into account.
- For each solution submitted, write in the table above how many pages you submit. **Do not submit scratchworks and solutions that are not to be graded**. Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.
- Please write big and legibly. Proofs cannot be graded unless they can be read. Justify your arguments with complete sentences. *Answers, even correct, without justifications will not receive (full) credit*.
- **If you make use of a result which you do not prove, then write the complete statement of the result you use separately underneath your proof, and refer to it within your proof.**
- You should not interpret any question as trivial by referring to a result from a textbook.

Part A. Group Theory

Questions.

(1) Let G be a *finite* group, H be a *proper* subgroup of G, and let N be a *normal* subgroup of G. Set $[G : N] = r$ and $|H| = m$. Assume $\gcd(r, m) = 1$.
Prove that $H \subseteq N$.

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(2) Let G be a group of order $2 \cdot 3^2 \cdot 5^2$.
Prove that G is *not* simple.
(Hint: consider an action of G on the normalizer of a Sylow subgroup of G.)

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(3) Let G_1 and G_2 be groups, and let $\psi : G_1 \to G_2$ be a group homomorphism.
Assume $G_1' = G_1$, where $G_1' = < \{xyx^{-1}y^{-1} : x, y \in G_1\} >$ is the *commutator* subgroup of G_1.
Assume further G_2 is *solvable*.
Prove that $\mathrm{im}(\psi) = \{e\}$, that is, the image of ψ is the trivial subgroup of G_2.

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Part B. Field and Galois Theory

Questions.

(4) Determine the *minimal polynomial* of $\alpha = \sqrt[6]{2}$ over the field $\mathbb{Q}(\sqrt{2})$.
Justify your work to receive full credit.

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(5) Let F be a field of *prime characteristic* p.
Assume the following condition holds: "whenever E/F is an *algebraic* field extension, it follows that E/F is *separable*, that is, the minimal polynomial of each element of E over F has no multiple roots in any field extension of F."
Let $\alpha \in F$. Prove that there exists an element $\beta \in F$ such that $\alpha = \beta^p$.
(Hint: start by considering the splitting field E of the polynomial $x^p - \alpha$ over F.)

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(6) Let α be a *real root* of the polynomial $p(x) = x^{13} - 5 \in \mathbb{Q}[x]$, and let $L = \mathbb{Q}(\alpha)$.
Use Galois Theory and prove that L is *not* contained in any *cyclotomic* field over $\mathbb{Q}$.

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Part C. Ring and Module Theory

Conventions.

- In this section, R denotes a commutative ring which has *multiplicative identity* 1 such that $1 \neq 0$.

Questions.

(7) Let $\mathfrak{q}$ be an ideal of R.
Assume $\sqrt{\mathfrak{q}}$ is a *maximal ideal* of R.
Prove $\mathfrak{q}$ is a *primary* ideal of R, that is, prove the following:
$\mathfrak{q} \neq R$, and whenever $a, b \in R$ such that $ab \in \mathfrak{q}$, one has $a \in \mathfrak{q}$ or $b \in \sqrt{\mathfrak{q}}$.

(Hint: recall that $\sqrt{\mathfrak{q}} = \{x \in R : x^n \in \mathfrak{q}$ for some integer $n \geq 1\}$ is the *radical* of $\mathfrak{q}$.
If needed, you may use the following fact without proof:
if I and J are ideals of R such that $J + \sqrt{I} = R$, then it follows that $I + J = R$.)

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(8) Assume R is an integral domain. Assume further R is *not* a field.
Prove that R is *not* an Artinian ring.

$* * *$

(9) Let $\mathfrak{p}$ be a *prime ideal* of R, and let M be a *finitely generated* (left) R-module.
Assume the localization of M at $\mathfrak{p}$ is zero, that is, $M_{\mathfrak{p}} = 0$.
Prove that there is an element $v \in \mathrm{Ann}_R(M)$ such that $v \notin \mathfrak{p}$.

(Recall that $\mathrm{Ann}_R(M) = \{r \in R : rx = 0$ for all $x \in M\}$ is the *annihilator* of M.)

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Write **big** and **legibly**. Your proofs cannot be graded if they cannot be read. Justify your arguments.

page # of problem # | name: