Fall 2021

Analysis

WVU Mathematics Department

Analysis exam Fall 2021

1. Suppose μ is the measure on a measurable space $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$ defined by

$$\mu(E) = \sum_{n \in E} \frac{1}{2^n}.$$

Prove that for every $\varepsilon > 0$, there exists a set $E \subset \mathbb{Z}^+$ with $\mu(\mathbb{Z}^+ \setminus E) < \varepsilon$ such that $f_1, f_2, \cdots$ converges uniformly on E for every sequence of functions $f_1, f_2, \cdots$ from $\mathbb{Z}^+$ to $\mathbb{R}$ that converges pointwise on $\mathbb{Z}^+$.

Remark: this problem is to test students' understanding of uniform convergence and measures. For any ε one can take N large enough such that $\sum\limits_{n>N} \frac{1}{2^n} < \varepsilon$. Then $E = \{1, 2, \cdots N\}$ satisfy the condition because E is a finite set.

2. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \cdots$ are $\mathcal{S}$-measurable functions from X to $\mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} \int |f_k| d\mu < \infty$. Prove that there exists $E \in S$ such that $\mu(X \setminus E) = 0$ and $\lim\limits_{k \to \infty} f_k(x) = 0$ for every $x \in E$.

Remark: Application of Monotone convergence theorem. $\int \sum\limits_{k=1}^{\infty} |f_k| d\mu = \sum\limits_{k=1}^{\infty} \int |f_k| d\mu < \infty$. Thus $\sum\limits_{k=1}^{\infty} |f_k|$ converges a.e..

3. Suppose $b_1, b_2, \cdots$ is a sequence of real numbers. Define $f : \mathbb{R} \to [0, \infty]$ by

$$f(x) = \begin{cases} \sum\limits_{k=1}^{\infty} \frac{1}{4^k |x - b_k|}, & \text{if } x \notin \{b_1, b_2, \cdots\}, \\ \infty, & \text{if } x \in \{b_1, b_2, \cdots\}. \end{cases}$$

Prove that $|\{x \in \mathbb{R} : f(x) < 1\}| = \infty$.

Remark: This is to test students' understanding on measures. construct $B = \bigcup\limits_{k=1}^{\infty} \{x \in \mathbb{R} : |x - b_k| < \frac{1}{2^k}\}$. Then on B^c, $f < 1$.

4. Suppose $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{t \to 0+} \frac{1}{2t} \int_{b-t}^{b+t} |f - f_{[b-t, b+t]}| = 0$$

for almost every $b \in \mathbb{R}$. Here $f_I = \frac{1}{|I|} \int_I f$ is the the mean of the function f on the interval I.

Remark: this is to test Lebesgue differentiation theorem. $\frac{1}{2t} \int_{b-t}^{b+t} |f - f_{[b-t,b+t]}| = \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b) + f(b) - f_{[b-t,b+t]}| \leq \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| + \frac{1}{2t} \int_{b-t}^{b+t} |f(b) - f_{[b-t,b+t]}| \leq \frac{1}{t} \int_{b-t}^{b+t} |f - f(b)|$. Then using Lebesgue differentiation theorem.