

Fall 2022

## Analysis

WVU Mathematics Department

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# Analysis exam Fall 2022

Solve exactly 6 out of 8 problems below

August 15, 2022

1. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and such that  $\lim_{x \rightarrow -\infty} f(x)$  and  $\lim_{x \rightarrow \infty} f(x)$  both exist as real numbers. Prove that  $f$  is uniformly continuous.

2. Let  $f$  and  $g$  be real-valued functions defined on the open interval  $(-1, 1)$ . Prove that if  $f, g$  are both continuous at 0, then so is the product  $fg$ . If  $f$  and  $fg$  are continuous at 0, does it follow that so is  $g$ ?

3. Let  $\{a_n\}_{n \geq 1}$  be a sequence of real numbers such that  $a_{n+1} = a_n^2 - 2a_n + 2$  for all integers  $n \geq 1$ . Prove that  $\{a_n\}_{n \geq 1}$  is convergent if and only if  $0 \leq a_1 \leq 2$ . What is its limit in this case?

4. A fixed point of a function  $f$  is a value  $x$  where  $f(x) = x$ . Show that if  $f$  is differentiable on an interval with  $f'(x) \neq 1$ , then  $f$  can have at most one fixed point.

5. Suppose  $\mathcal{S}$  is the smallest  $\sigma$ -algebra on  $\mathbb{R}$  containing  $\{(r, r + 1) : r \in \mathbb{Q}\}$ . Prove that  $\mathcal{S}$  is the collection of Borel subsets of  $\mathbb{R}$ . Here  $\mathbb{R}$  is the set of real numbers, and  $\mathbb{Q}$  is the set of rational numbers.

6. Suppose  $f \in L^1(\mathbb{R})$ . Denote  $f_t(x) = f(x - t)$ . Prove that

$$\lim_{t \rightarrow 0} \|f - f_t\|_{L^1} = 0.$$

7. Assume  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lebesgue measurable. The Hardy-Littlewood maximal function  $f^* : \mathbb{R} \rightarrow [0, \infty]$  is defined by

$$f^*(x) = \sup_{t > 0} \frac{1}{2t} \int_{x-t}^{x+t} |f|.$$

Prove that  $\{x \in \mathbb{R} \mid f^*(x) > c\}$  is an open subset of  $\mathbb{R}$  for every real number  $c$ .

8. (1) Prove that the function  $f : (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = \frac{\sin x}{x}$  is not Lebesgue integrable.

(2) Show that the limit  $\lim_{t \rightarrow \infty} \int_0^t f(x) dx$  exists.