Fall 2022

Analysis

WVU Mathematics Department

Analysis exam Fall 2022

Solve exactly 6 out of 8 problems below

August 15, 2022

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and such that $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ both exist as real numbers. Prove that f is uniformly continuous.

2. Let f and g be real-valued functions defined on the open interval $(-1, 1)$. Prove that if f, g are both continuous at 0, then so is the product fg. If f and fg are continuous at 0, does it follow that so is g?

3. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_{n+1} = a_n^2 - 2a_n + 2$ for all integers $n \geq 1$. Prove that $\{a_n\}_{n\geq 1}$ is convergent if and only if $0 \leq a_1 \leq 2$. What is its limit in this case?

4. A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

5. Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbb{R}$ containing $\{(r, r+1) : r \in \mathbb{Q}, \}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbb{R}$. Here $\mathbb{R}$ is the set of real numbers, and $\mathbb{Q}$ is the set of rational numbers.

6. Suppose $f \in L^1(\mathbb{R})$. Denote $f_t(x) = f(x - t)$. Prove that

$$\lim_{t\to 0} \|f - f_t\|_{L^1} = 0.$$

7. Assume $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. The Hardy-Littlewood maximal function $f^* : \mathbb{R} \to [0, \infty]$ is defined by

$$f^*(x) = \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |f|.$$

Prove that $\{x \in \mathbb{R} \mid f^*(x) > c\}$ is an open subset of $\mathbb{R}$ for every real number c.

8. (1) Prove that the function $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{\sin x}{x}$ is not Lebesgue integrable.
(2) Show that the limit $\lim_{t\to\infty} \int_0^t f(x)dx$ exists.