Fall 2021

Differential Equations

WVU Mathematics Department

ODE ENTRANCE EXAM, FALL 2021

September 1 2021

Solve all six problems. Show all work. Explain and justify your answers. All problems carry equal weight.

Name ______________________________ **Total Score** ________

1. Let $b > 0$ and assume that $f(x) \in C([0, b])$ satisfies

$$0 \leq f(x) - 1 \leq \int_0^x [\sin(\sqrt{f(s) - 1})]^2 ds, \quad \text{for all } x \in [0, b].$$

Prove that $f(x)$ is constant on $[0, b]$.

2. Prove that all real valued solutions to the system

$$y_1' = \cos(y_1 y_2), \qquad y_2' = \frac{2y_1 y_2}{1 + y_1^2 + y_2^2}$$

exist on $(-\infty, \infty)$.

3. Determine all real valued column vector solutions $y(t)$ to the system

$$y' = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} y - \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

that are bounded on the interval $(-\infty, 0]$. Does $\lim_{t \to -\infty} y(t)$ of these bounded solution exist? If yes find that limit.

4. Consider the initial value problem

$$\frac{dx}{dt} = t^2 + x^2, \qquad x(0) = 1 \tag{1}$$

a. Show that (1) has a solution defined for $t \in [0, 1/5]$.

b. Show that (1) does not have a solution defined for $t \in [0, 5]$.

5. Consider the ODE system

$$\begin{aligned} \frac{dx}{dt} &= x(1-y) \\ \frac{dy}{dt} &= y(x-1) \end{aligned} \tag{2}$$

a. Determine the critical points of (2) and their stability.
b. Suppose $(x(t), y(t))$ is a solution of the system with $(x(0), y(0)) \in \mathbb{R}^2_{\geq 0}$. Show that $(x(t), y(t)) \in \mathbb{R}^2_{\geq 0}$ for all $t \geq 0$ where the solution is defined.
c. Show that the solution $(x(t), y(t))$ of (2) with $(x(0), y(0)) = (2, 1/2)$ satisfies

$$x(t) + y(t) - \log(x(t)y(t)) = 5/2.$$

(Hint. Show that the expression is constant with respect to t.)
d. Show that (2) has periodic solutions.

6. Find a fundamental matrix of the system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$