Spring 2021

Differential Equations

WVU Mathematics Department

ODE ENTRANCE EXAM, SPRING 2021

April 21 2021

Solve all six problems. Show all work. Explain and justify your answers. All problems carry equal weight.

Name _____ **Total Score** _____

1 Let $\alpha \in \mathbb{R}$ and let $x(t) = [x_1(t), x_2(t), x_3(t)]$ denote the solution of the initial value problem $dy/dt = Ay,\ x(0) = [1, 1, \alpha]$, where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

a. Find $x(t)$.

b. Find $\lim_{t\to\infty} \dfrac{x_1(t)}{x_2(t)}$.

2 Show that the initial value problem

$$\frac{dx}{dt} = t^2 + x^2, \qquad x(0) = 0 \tag{1}$$

has a unique solution defined on an interval for t containing $(-\sqrt{2}/2, \sqrt{2}/2)$.

3 a. Find all equilibrium points of the system

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

and study their stability.

b. Sketch a phase diagram with arrows indicating the direction of increasing time.

4 Consider the ODE system

$$\dot{x} = -x^3 + xy^2$$
$$\dot{y} = -x^2y - y^3$$

Show that $(0,0)$ is a globally asymptotically stable steady state. *Hint: Look for a quadratic Lyapunov function for the system.*

5 Show that the system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x + (1 - x^2 - y^2)y\end{aligned}$$

has a periodic solution.

6 Find the general solution of

$$y'' + 4y = 8\tan t,\ \ t \in (-\pi/2, \pi/2).$$