Fall 2022

Differential Equations

WVU Mathematics Department

ODE ENTRANCE EXAM, FALL 2022

August 18 2022

Solve all six problems. Show all work. Explain and justify your answers. All problems carry equal weight. Notation: $\frac{dy}{dt} = y'$.

Name ________________________________ **Total Score** ________

1. Let $y = f(t)$ be a continuous non constant periodic function on $\mathbb{R}$ with period $P > 0$. Show that:
a) $y = f(t)$ is bounded on $\mathbb{R}$;
b) $\lim_{t\to\infty} f(t)$ does not exist;
c) no vector solution to any initial value problem of the system

$$y_1' = y_1^2 y_2^4 + 1,\ y_2' = \sin(y_1^2 y_2^4) + 1 \tag{1}$$

can be periodic.

2. Estimate an interval of existence for the scalar initial value problem

$$z'' = \frac{z+1}{2-z} + tz' + 2,\ z(0) = 1, z'(0) = 2. \tag{2}$$

3. Consider the linear system

$$y' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} y - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3}$$

a) Determine the critical points of (3).
b) Determine all vector solutions of (3) that satisfy $\lim_{t\to\infty} |y(t)| = \infty$ where $|.|$ is a suitable norm.

4. Let $\phi(t)$ denote a solution of the initial value problem

$$\begin{aligned} y' &= (y-1)(y-2)\dots(y-2022) \\ y(0) &= \pi. \end{aligned}$$

Show that ϕ defined for $t \in \mathbb{R}$ and find $\lim_{t\to\infty} \phi(t)$ and $\lim_{t\to-\infty} \phi(t)$.

5. a) State an existence and uniqueness theorem for first-order ODE systems.
b) Find $t_0, y_0 \in \mathbb{R}$ such that the initial value problem

$$y' = y^{2/3}, \; y(t_0) = y_0$$

does not have unique solution. Why does the theorem you stated in a) fail to apply?

6. Let A be a 2×2 real matrix, and suppose the linear ODE system

$$y' = Ay$$

has solutions ϕ_1, ϕ_2, ϕ_3 satisfying

$$\phi_1(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; \phi_2(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; \phi_3(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

If additionally

$$\phi_1(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; \phi_2(1) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

find $\phi_3(1)$.