

Spring 2022

Differential Equations

WVU Mathematics Department

Follow this and additional works at: <https://researchrepository.wvu.edu/math-grad-exams>

Recommended Citation

WVU Mathematics Department, "Differential Equations" (2022). *M.S. Advanced and Ph.D. Entrance Exams*. 41.

<https://researchrepository.wvu.edu/math-grad-exams/41>

This Other is brought to you for free and open access by the Mathematics at The Research Repository @ WVU. It has been accepted for inclusion in M.S. Advanced and Ph.D. Entrance Exams by an authorized administrator of The Research Repository @ WVU. For more information, please contact beau.smith@mail.wvu.edu.

ODE ENTRANCE EXAM, SPRING 2022

April 27 2022

Solve all six problems. Show all work. Explain and justify your answers. All problems carry equal weight. Notation: $\frac{dy}{dt} = y'$.

Name _____

Total Score _____

1. Consider the second order scalar ordinary differential equation

$$y'' + 4y^3 = 0. \quad (1)$$

a) Show that if (1) has a solution $y(t)$ on \mathbb{R} then $y(t)$ and $y'(t)$ are bounded on \mathbb{R} .

b) Let $y(t)$ be a solution of (1) with initial conditions $y(0) = \alpha \in \mathbb{R}$, $y'(0) = \beta \in \mathbb{R}$.

Show that y is defined on \mathbb{R} . Find $\min_{t \in \mathbb{R}} y(t)$ and $\max_{t \in \mathbb{R}} y(t)$.

c) If $y(t)$ is the solution of the IVP in b), determine bounds for $y''(t)$, $y'''(t)$ on \mathbb{R} .

d) Does (1) possess any non constant periodic solution? Why?

2. Given the vectorial system of equations

$$y' = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} y - \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

determine:

a) All of its bounded solutions on \mathbb{R} .

b) All of its bounded solutions on $[0, \infty)$.

3. Explain the essence of the method of successive approximations for systems of ordinary differential equations

$$y' = f(t, y). \quad (2)$$

Formulate a theorem that guarantees existence and uniqueness of solutions to an initial value problem of (2).

4. Consider a solution $\phi(t)$ the initial value problem

$$y' = \frac{1}{t^2 + y^2}, \quad y(1) = 1 \quad (3)$$

defined on $[1, \infty)$.

- a) Show that $\phi(t) - \arctan(t)$ is decreasing on $[1, \infty)$.
- b) Show that $L = \lim_{t \rightarrow \infty} \phi(t)$ exists and $L \leq 1 + \frac{\pi}{4}$.

5. Let ϕ_1 and ϕ_2 be two solutions of

$$y'(t) = t^2 + 2 \sin(3y) \tag{4}$$

defined on an open interval I containing 0.

- a) Show that $|\phi_1(t) - \phi_2(t)| \leq |\phi_1(0) - \phi_2(0)| + 6 \int_0^t |\phi_1(s) - \phi_2(s)| ds$ for all $t \in I \cap [0, \infty)$.
- b) Show that $|\phi_1(t) - \phi_2(t)| \leq |\phi_1(0) - \phi_2(0)| e^{6t}$ for all $t \in I \cap [0, \infty)$.

6. Consider the system of ODEs

$$x' = x^2y - x, \quad y' = -x^2y + x. \tag{5}$$

- a) Show that for any solution $(x(t), y(t))$ of (5) defined on an interval $[0, a)$ there exists a real number T such that $x(t) + y(t) = T$ for all $t \in [0, a)$.
- b) Draw a phase portrait for (5) containing the direction of the vector field and typical trajectories.
- c) Show that any solution of (5) with positive initial conditions is defined on $[0, \infty)$.
- d) Find $\lim_{t \rightarrow \infty} x(t)$ for solutions $(x(t), y(t))$ with each of the following initial conditions: $(x(0), y(0)) = (1/4, 1/4)$; $(x(0), y(0)) = (1/2, 2)$; $(x(0), y(0)) = (1, 2)$.