Spring 2022

Differential Equations

WVU Mathematics Department

ODE ENTRANCE EXAM, SPRING 2022

April 27 2022

Solve all six problems. Show all work. Explain and justify your answers. All problems carry equal weight. Notation: $\frac{dy}{dt} = y'$.

Name ______________________________ **Total Score** ________

1. Consider the second order scalar ordinary differential equation

$$y'' + 4y^3 = 0. \tag{1}$$

a) Show that if (1) has a solution $y(t)$ on $\mathbb{R}$ then $y(t)$ and $y'(t)$ are bounded on $\mathbb{R}$.

b) Let $y(t)$ be a solution of (1) with initial conditions $y(0) = \alpha \in \mathbb{R}$, $y'(0) = \beta \in \mathbb{R}$. Show that y is defined on $\mathbb{R}$. Find $\min_{t\in\mathbb{R}} y(t)$ and $\max_{t\in\mathbb{R}} y(t)$.

c) If $y(t)$ is the solution of the IVP in b), determine bounds for $y''(t)$, $y'''(t)$ on $\mathbb{R}$.

d) Does (1) possess any non constant periodic solution? Why?

2. Given the vectorial system of equations

$$y' = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} y - \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

determine:

a) All of its bounded solutions on $\mathbb{R}$.

b) All of its bounded solutions on $[0, \infty)$.

3. Explain the essence of the method of successive approximations for systems of ordinary differential equations

$$y' = f(t, y). \tag{2}$$

Formulate a theorem that guarantees existence and uniqueness of solutions to an initial value problem of (2).

4. Consider a solution $\phi(t)$ the initial value problem

$$y' = \frac{1}{t^2 + y^2}, \quad y(1) = 1 \tag{3}$$

defined on $[1, \infty)$.

a) Show that $\phi(t) - \arctan(t)$ is decreasing on $[1, \infty)$.
b) Show that $L = \lim_{t\to\infty} \phi(t)$ exists and $L \leq 1 + \frac{\pi}{4}$.

5. Let ϕ_1 and ϕ_2 be two solutions of

$$y'(t) = t^2 + 2\sin(3y) \tag{4}$$

defined on an open interval I containing 0.
a) Show that $|\phi_1(t) - \phi_2(t) \leq |\phi_1(0) - \phi_2(0)| + 6\int_0^t |\phi_1(s) - \phi_2(s)|ds$ for all $t \in I \cap [0, \infty)$.
b) Show that $|\phi_1(t) - \phi_2(t)| \leq |\phi_1(0) - \phi_2(0)|e^{6t}$ for all $t \in I \cap [0, \infty)$.

6. Consider the system of ODEs

$$x' = x^2y - x, \quad y' = -x^2y + x. \tag{5}$$

a) Show that for any solution $(x(t), y(t))$ of (5) defined on an interval $[0, a)$ there exists a real number T such that $x(t) + y(t) = T$ for all $t \in [0, a)$.

b) Draw a phase portrait for (5) containing the direction of the vector field and typical trajectories.

c) Show that any solution of (5) with positive initial conditions is defined on $[0, \infty)$.

d) Find $\lim_{t\to\infty} x(t)$ for solutions $(x(t), y(t))$ with each of the following initial conditions: $(x(0), y(0)) = (1/4, 1/4)$; $(x(0), y(0)) = (1/2, 2)$; $(x(0), y(0)) = (1, 2)$.