Fall 2021

Topology

WVU Mathematics Department

NAME (print): ____________________

Topology Ph.D. Entrance Exam, September 2021

Solve ***four*** from the following five exercises. *We will grade only four of them, so you need to decide which four solutions to submit.* Write a solution of each exercise on a separate page. After you finish, you will need to take pictures of your work and sent it to the examiners according to the separate instructions.

Ex. 1. Let $\langle X, \tau \rangle$ be a topological space.

(a) Finish the following definitions:

- $\langle X, \tau \rangle$ is a Hausdorff (or T_2) topological space provided ...
- $\langle X, \tau \rangle$ is aT_1 topological space provided ...

(b) Show that if X is a finite and T_1, then it is Hausdorff.

(c) Give an example of infinite T_1 topological space that is not Hausdorff. Explain briefly why your example has the desired properties.

Ex. 2. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\langle A_n : n \in \mathbb{N} \rangle$ be a sequence of nonempty subsets of a topological space X such that $A_n \cup A_{n+1}$ is connected for every $n \in \mathbb{N}$. Show that $A := \bigcup_{n=1}^{\infty} A_n$ is a connected subset of X.

Do not assume that any $\mathcal{A}_n$ is connected or that $A_n \cap A_{n+1} \neq \emptyset$ for some distinct $n, m \in \mathbb{N}$.

Ex. 3. For a nonempty set $S \subset \mathbb{R}$ let $\mathcal{B}_S := \{[a, b) : a < b \text{ and } a, b \in S\}$. Assume that $S, T \subset \mathbb{R}$ are such that the families $\mathcal{B}_S$ and $\mathcal{B}_T$ are the bases for the topologies $\mathcal{T}_S$ and $\mathcal{T}_T$, respectively. Show that if $\mathcal{T}(\mathcal{B}_S)$ is finer than $\mathcal{T}(\mathcal{B}_T)$ (i.e. $\mathcal{T}(\mathcal{B}_T) \subseteq \mathcal{T}(\mathcal{B}_S)$), then $T \subset S$.

Ex. 4. Recall that a topological space X is *Lindelöf* provided every open cover of X contains a countable subcover. Show, using only this definition, that every Lindelöf metric space $\langle X, d \rangle$ has a countable basis. Include the definition of the basis and the argument that the collection that you constructed is actually a basis for X.

Ex. 5. Consider $\mathbb{R}^\omega$ with the product topology, where $\omega = \{0, 1, 2, \ldots\}$. For every $n \in \omega$ let $\pi_n \colon \mathbb{R}^\omega \to \mathbb{R}$ be the projection onto nth coordinate, that is, $\pi_n(x) = x(n)$ for every $x \in \mathbb{R}^\omega$.

(a) Show that if $K \subset \mathbb{R}^\omega$ is compact, then for every $n \in \omega$ the image $\pi_n[K]$ is a bounded subset of $\mathbb{R}$.

(b) Use part (a) to show that $\mathbb{R}^\omega$ considered with the product topology is not a countable union of compact sets.