Spring 2021

Topology

WVU Mathematics Department

NAME (print): ______________________________

Topology Ph.D. Entrance Exam, April 2021

Solve ***four*** from the following five exercises. *We will grade only four of them, so you need to decide which four solutions to submit.* Write a solution of each exercise on a separate page. After you finish, you will need to take pictures of your work and sent it to the examiners according to the separate instructions.

In what follows the symbol $\mathrm{cl}(A)$ stands for the closure of A. Any subset of $\mathbb{R}$ is considered with the standard topology, unless stated otherwise.

Ex. 1. Define the following topological notions.

(i) Compactness of a topological space $\langle X, \mathcal{T} \rangle$.

(ii) Separability of a topological space $\langle X, \mathcal{T} \rangle$.

(iii) A metric topology $\mathcal{T}_d$ of a metric space $\langle X, d \rangle$.

(iv) A metrizable topological space $\langle X, \mathcal{T} \rangle$.

Then, using only provided definitions, prove that every compact metrizable space $\langle X, \mathcal{T} \rangle$ is separable.

Ex. 2. Give a standard definition of a *normal* topological space. Then, using only provided definition, show that for every normal topological space X:

- for every closed disjoint sets $A, B \subset X$ there exist open sets $U \supset A$ and $V \supset B$ with disjoint closures.

Ex. 3. Let $\langle X, d \rangle$ be a compact metric space containing more than one point. Let $f \colon X \to X$ be a continuous surjection. Show that there are distinct $x, y \in X$ such that $d(f(x), f(y)) \geq d(x, y)$. **Hint:** Use, without a proof, the fact that $d \colon X \times X \to \mathbb{R}$ is continuous.

Ex. 4. Let X be an arbitrary set, Y be a topological space, and $f \colon X \to Y$ be an arbitrary function. Let $\{A_n \colon n \in \mathbb{N}\}$ be such that $f[A_n]$ is connected for every $n \in \mathbb{N}$. Show that if $A_n \cap A_{n+1} \neq \emptyset$ for every $n \in \mathbb{N}$, then $f[\bigcup_{n \in \mathbb{N}} A_n]$ is connected.

Ex. 5. Let X be a compact space, $x \in X$, and $\mathcal{U}$ be a family of open subset of X such that

- $x \in U$ for every $U \in \mathcal{U}$, and
- $U \cap V \in \mathcal{U}$ for all $U, V \in \mathcal{U}$.

Show that if $\bigcap\{\mathrm{cl}(U) \colon x \in U \in \mathcal{U}\} = \{x\}$, then $\mathcal{U}$ is a (neighborhood) basis at x for X. Include the definition of a (neighborhood) basis at x for X.