Spring 2022

Topology

WVU Mathematics Department

NAME (print): ______________________________

Topology Ph.D. Entrance Exam, April, 2022

Solve ***four*** from the following five exercises. *We will grade only four of them, so you need to decide which four solutions to submit.* Write a solution of each exercise on a separate page.

In what follows the symbols $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ stand, respectively, for the interior and the closure of A. Any subset of $\mathbb{R}$ is considered with the standard topology, unless stated otherwise.

Ex. 1. Let X be a topological space. Prove that $A \subset X$ is closed if, and only if, for every $x \in X \setminus A$ there exists closed subset F of X such that $A \subset F$ and $x \notin F$.

Ex. 2. Show that every infinite connected compact Hausdorff space is uncountable.

Ex. 3. Consider $\mathbb{R}^n$ with the standard topology. Show, that a nonempty $A \subset \mathbb{R}^n$ is compact if, and only if, $f[A]$ is bounded for every continuous $f\colon A \to \mathbb{R}$.

Ex. 4. Let $\mathbb{R}^\omega$ be a countable product of real lines and consider it with the product topology. Show that

$$Z := \{(x_n)_n \in \mathbb{R}^\omega \colon x_n \text{ is rational for all but finitely many } n\},$$

considered with the subspace topology of $\mathbb{R}^\omega$, is connected.

Ex. 5. Let $\langle X, d\rangle$ be a metric space. For any $x_0 \in X$ and $\varepsilon > 0$ define

$$B(x_0, \varepsilon) := \{x \in X \colon d(x, x_0) < \varepsilon\} \quad \& \quad B[x_0, \varepsilon] := \{x \in X \colon d(x, x_0 \leq \varepsilon\}.$$

(i) Show that $\mathrm{cl}(B(x_0, \varepsilon)) \subseteq B[x_0, \varepsilon]$.

(ii) Give an example of a metric space X where for some $x_0 \in X$ and $\varepsilon > 0$ we have $\mathrm{cl}(B(x_0, \varepsilon) \neq B[x_0, \varepsilon]$.