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Multi-selectible continua

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(ABSTRACT)

A particular coexistence of multiple continuous selections (a "multi-selection") within a function space for a non-degenerate metric continuum X is investigated for the hyperspaces 2^X and $C(X)$. It is demonstrated that (i) multi-selections do not exist for 2^X and that (ii) the existence of a multi-selection for $C(X)$ is equivalent to X being a dendrite.

Dedication

The author wishes to express his gratitude to his wife, Valerie, to his daughter Samantha, and to his advisor, Dr. Sam B. Nadler, Jr. Whether the contribution was in time, in patience and understanding, or in knowledge and encouragement, each has played an important part and made personal sacrifice in seeing this work to its final state.

This research is dedicated to them.

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0 Introduction

Let M be a metric space, 2^M denote the collection of all closed subsets of M with the Vietoris topology, and let $\mathcal{F}$ be a subspace of 2^M. Then a *selection for* $\mathcal{F}$ is a continuous function $\sigma : \mathcal{F} \to M$ which has the following property:

$$\sigma(F) \in F, \forall F \in \mathcal{F}$$

A selection, then, is readily seen to be a shifting of the more general notion of a "choice function" to the setting of hyperspaces of sets.

The study of selections is not new, but rather begins with the prominent early work of Michael in 1951. Early investigations of research mathematicians were focused upon answering the natural question, "for what spaces to selections exist?" The question is far from trivial; while volumes of work in this area have been produced with significant results, the question remains largely unanswered and research continues today. [For a history of the early work in selections (including references to much of Michael's work), the reader is referred to [6]].

In this paper, we will not be so concerned with *when* a hyperspace admits a selection; rather we will address the question of *how many* selections a space may admit, and when different selections for a space may "coexist" within a particular function space.

Little work has been accomplished to date in answering this question of the number of selections, though the question is not new. In a recent paper by Nogura and Shakhmatov [9], the arc is characterized as an infinite, separable, connected, Hausdorff space *which admits exactly two selections* (where a selection is specifically defined for our $\mathcal{F} = 2^M$).

So we suppose a space M does admit a selection from the hyperspace $\mathcal{F}$. We ask under what circumstances may we continuously vary the selection function itself, thus producing uncountably many selections. With M a non-degenerate continuum, we will examine the maximal case wherein we will homeomorphically embed the hyperspace $\mathcal{F}$ into a function space in such a way that for, any fixed k in the domain, evaluation of each function in the space at k will yield a selection. Necessary and sufficient conditions upon M and $\mathcal{F}$ will be determined which will permit such a construction.

1 Preliminaries:

A *continuum* is a compact, connected metric space. A *dendroid* is a hereditarily unicoherent, arc-wise connected continuum. A *dendrite* is a locally connected dendroid.

Unless otherwise defined,

1. X will denote a continuum with metric d
2. $C_{1/3}$ will denote the Cantor Middle-Thirds Set
3. 2^X will denote the space of (non-empty) compact subsets of X
4. $C(X)$ will denote the space of (non-empty) subcontinua of X

5. H_d will denote the Hausdorff metric on the non-empty compact subsets of X derived from metric d

6. For metric compactum A, X^A will denote the set of all continuous functions from A into X

7. ρ will be the "supremum" metric for the function spaces in question

8. I will denote the closed unit interval with the ordinary metric.

9. for each $n \in \mathcal{N}$, $F_n(X)$ will denote the space of all n-point subsets of X, considered as a subspace of 2^X

Let $\mathcal{H} \subset 2^X$. A continuous function $\sigma : \mathcal{H} \to X$ is said to be a *(continuous) selection* if, for every $A \in \mathcal{H}$, $\sigma(A) \in A$. A continuum X is said to be *selectible* if there exists a continuous selection from $\mathcal{H}$ into X, where $\mathcal{H} \subset 2^X$ is given from the context (generally, $\mathcal{H}$ will be either 2^X or $C(X)$).

Lastly, all functions are considered continuous unless explicitly stated to the contrary.

Theorem 1.1 (Michael [5]) *Let*

$$\Psi : X^{C_{1/3}} \longrightarrow 2^X$$

be defined pointwise as follows

$$f \longmapsto Rng(f)$$

Then the map Ψ is a continuous, open surjection.

Henceforth, all references to the function Ψ will refer to to Michael's Ψ function of this theorem.

Theorem 1.2 (Evaluation) *Let $\mathcal{H}$ be a subset of 2^X, and suppose there exists a continuous function*

$$\Sigma : \mathcal{H} \longrightarrow X^{C_{1/3}}$$

such that

$$\Psi \circ \Sigma = i_{\mathcal{H}}$$

[That is, Σ is a right inverse for Ψ on $\mathcal{H}$.] Then, for each k in $C_{1/3}$, the map

$$\sigma_k : \mathcal{H} \longrightarrow X$$

given by

$$\sigma_k(A) = [\Sigma(A)](k)$$

is a continuous selection.

Proof:

Let X be a continuum with metric d. Let $k \in C_{1/3}$, and functions Σ and σ_k be as defined above. Let $A \in \mathcal{H}$ and let $\epsilon > 0$. By the continuity of Σ, there exists a $\delta > 0$ such that for every $B \in \mathcal{H}$, if $H_d(A,B) < \delta$ then $\rho(\Sigma(A), \Sigma(B)) < \epsilon$. Since $\Psi \circ \Sigma = i_{\mathcal{H}}$, Σ is necessarily one-to-one. By definition of Ψ, $\sigma_k(A) = [\Sigma(A)](k)$ (that is, the function $\Sigma(A)$ evaluated at k) is an element of A. Similarly, $\sigma_k(B) \in B$. As ρ is the metric of uniform continuity, we have that $d(\sigma_k(A), \sigma_k(B)) < \epsilon$ as well. Thus, function σ_k is a continuous selection from $\mathcal{H}$ into X.$\square$

Definition 1.1 (Multi-Selection) *Let X be a topological space, $\mathcal{H}$ be a subspace of the hyperspace 2^X and let A be a compactum. Then a map*

$$\Sigma : \mathcal{H} \longrightarrow X^A$$

that is a right inverse for Ψ (with respect to the identity map on $\mathcal{H}$) will be called an $(\mathcal{H}, A)$ multi-selection for X*. Since context will often dictate the underlying spaces in question, we sometimes simply refer to Σ as a multi-selection.*

Note that a multi-selection for X is an embedding of the hyperspace into the function space for which Michael's Ψ function is an R-map in the sense of Borsuk ([1], p. 7).

Corollary 1.1 *A necessary condition for the existence of a $(\mathcal{H}, C_{1/3})$ multi-selection for X is that there exists a continuous selection $\sigma : \mathcal{H} \rightarrow X$.*

2 Non-Existence of a $(2^X, C_{1/3})$ Multi-Selection

We begin our discussion of multi-selections by considering the most general of the hyperspaces ordinarily studied: the space 2^X. We establish immediately, however, that the structure of 2^X is in some sense too loose to support the existence of a multi-selection. We consider the class of $(2^X, C_{1/3})$ multi-selections since there are continuous functions from $C_{1/3}$ onto any sub-compactum of a continuum X (7.7 of [7], p. 106). By establishing that there is no such multi-selection for a (non-degenerate) continuum X, we will know by use of the domain transfer lemma (Lemma 3.1) that there is no $(2^X, A)$ multi-selection for X when A is any metric compactum.

The following theorem is a special case of [6, Theorem (5.3)] wherein the result is proven for $F_2(X)$ rather than 2^X:

Theorem 2.1 (Michael; Kuratowski, Nadler, Young) *Let X be a non-degenerate continuum. Then there exists a continuous selection $\sigma : 2^X \to X$ if, and only if, X is an arc.*

Theorem 2.2 *Let $\mathcal{H}$ be a subset of 2^I such that $\mathcal{H}$ contains $F_2(I) \cup F_3(I)$. Then there does not exist an $(\mathcal{H}, C_{1/3})$ multi-selection for I.*

PROOF:

Suppose there does exist such a multi-selection $\Sigma : \mathcal{H} \to I^{C_{1/3}}$. Let $T = \{0, \frac{1}{2}, 1\} \in \mathcal{H}$. Since $T \in \mathcal{H}$, by supposition there exists a selection $\tau = \Sigma(T) : \mathcal{H} \to I$. Let $A_0 = \tau^{-1}(0)$, $A_{\frac{1}{2}} = \tau^{-1}(\frac{1}{2})$, and $A_1 = \tau^{-1}(1)$. Since τ is continuous and maps onto T, these three subsets of $C_{1/3}$ are pair-wise disjoint and their union is all of $C_{1/3}$.

Let $S = \{a, b, c\}$ with the discrete metric. Define a homotopy

$$H : S \times I \longrightarrow I$$

as follows: For all t in I,

1. $H(a,t) = 0$
2. $H(c,t) = 1$
3. $H(b,t) = (1-t)/2$

[In effect, at time zero we begin with set T, and we subsequently move the central point toward 0, ending with the doubleton $\{0,1\} \in \mathcal{H}$.]

For all $t \in I$, we have that the image $H(S,t)$ is a compact subset of I, and thus an element of 2^I. Now, for all $t > 0$, note that we must have that

1. $[\Sigma(H(S,t))](A_0) = H(\{a\},t) = \{0\}$
2. $[\Sigma(H(S,t))](A_1) = H(\{c\},t) = \{1\}$
3. $[\Sigma(H(S,t))](A_{\frac{1}{2}}) = \{\, H(\{b\},t)\,\}$

Therefore, by continuity of both Σ and H, we must have the following:

1. $[\Sigma(\{0,1\})](A_0 \cup A_{\frac{1}{2}}) = \{0\}$
2. $[\Sigma(\{0,1\})](A_1) = \{1\}$

Symmetrically, we may argue with homotopy H' which will move the central point toward toward the point 1 (*i.e.*, $H'(b,t) = \frac{t+1}{2}$), resulting in the following:

1. $[\Sigma(\{0,1\})](A_0) = \{0\}$
2. $[\Sigma(\{0,1\})](A_{\frac{1}{2}} \cup A_1) = \{1\}$

This, however, contradicts the unique selection by Σ for the doubleton $\{0,1\}$ (an element of $\mathcal{H}$). Thus, we have proved our theorem.$\Box$

Corollary 2.1 *There does not exist a* $(2^I, C_{1/3})$ *multi-selection for* I*.*

Theorem 2.3 *There does not exist a* $(2^X, C_{1/3})$ *multi-selection for any non-degenerate continuum* X*.*

Proof of Theorem:

For the stated multi-selection to exist, we first must have the existence of an ordinary selection $\sigma : 2^X \to X$. This may occur only if X is an arc. As just proved, however, the arc has no such multi-selection.$\Box$

3 Multi-Selections for the space $C(X)$

We know consider the other widely studied hyperspace, $C(X)$. We establish immediately that simply restricting our hyperspace $\mathcal{H}$ to $C(X)$ is still not enough to ensure that there exists a $(C(X), C_{1/3})$ multi-selection for X, even when there does exist an ordinary selection $\sigma : C(X) \to X$.

The following observation pin-points one difficulty in establishing the existence of a multi-selection. What will follow in our main characterization theorem will demonstrate that that this obstacle is the *only* such obstacle.

Observation:

The existence of a continuous selection $\sigma : C(X) \to X$ is not sufficient to ensure the existence of a $(C(X), C_{1/3})$ multi-selection.

Example: Let F be the "Harmonic Fan" constructed in the plane as follows:

Let $S = \{1/2^n | n = 1, 2, \ldots\} \cup \{0\}$. Then

$$F = \bigcup_{x \in S} \{\overline{OP} \mid O = (0,0) \text{ and } P = (1, x)\}$$

(where $\overline{OP}$ represents the convex arc between points O and P). Let $L = \{(x, 0) | 0 \leq x \leq 1\}$, the "limiting arc" of F. It is known that there does exist a selection $\sigma : C(F) \to F$, and that, for any such selection, $\sigma(L)$ necessarily is the ramification point of F (the origin, O, in our construction) (*c.f.,* [8]).

Recall that a multi-selection of the form $\Sigma : C(F) \to F^{C_{1/3}}$ will necessarily assign to L a function $\gamma \in F^{C_{1/3}}$ with the property that γ maps $C_{1/3}$ onto L.

Therefore, there exists a k in $C_{1/3}$ such that $\gamma(k)$ is not the origin. By our evaluation theorem, evaluation of $\Sigma(A)$ at k (for each $A \in C(F)$) would be a selection σ — however, $\sigma(L)$ would not be the origin.

Therefore, there is no multi-selection for $C(F)$.$\square$

We have shown that, for any (non-degenerate) continuum X, there is no $(2^X, C_{1/3})$ multi-selection. The example above shows that there is no multi-selection for $C(F)$ in spite of the existence of an ordinary selection. Hence, the existence of a multi-selection may well seem rather dismal. However, there are continua for which multi-selections exist. The arc I is an easy example:

Theorem 3.1 *There exists a $(C(I), I)$ multi-selection for I.*

Proof:

For each $C \in C(I)$, define $\lambda_C : I \to C$ as the order-preserving linear map of I onto C. That is, for all $t \in I$,

$$\lambda_C(t) = [\max(C) - \min(C)]t + \min(C)$$

Now, define $\Sigma : C(I) \to I^I$ by the assignment $\Sigma(C) = \lambda_C$ for each $C \in C(I)$.

Clearly function Σ satisfies $\Psi \circ \Sigma = i$. It remains to be shown that Σ is continuous. Observe that for $A, B \in C(I)$,

$$H_d(A, B) = \max\{|\max(A) - \max(B)|, |\min(A) - \min(B)|\}$$

For convenience of notation, we assign the following values: $m_A = \min A,\ M_A = \max A,\ m_B = \min B,\ M_B = \max B$.

Let $\epsilon > 0$, and suppose that $H_d(A, B) < \epsilon/3$. Let $t_m \in I$ be the value which maximizes the expression

$$|\lambda_A(t_m) - \lambda_B(t_m)|$$

Then we have the following:

$$\begin{aligned}
\rho(\Sigma(A), \Sigma(B)) \quad &= \rho(\lambda_A, \lambda_B) \\
&= |\lambda_A(t_m) - \lambda_B(t_m)| \\
&= |\{(M_A - m_A)t_m + m_A\} - \{(M_B - m_B)t_m + m_B\}| \\
&\leq |(M_A - M_B)t_m + (m_B - m_A)t_m| + |m_A - m_B| \\
&\leq |M_A - M_B||t_m| + |m_B - m_A||t_m| + |m_A - m_B| \\
&\leq \tfrac{\epsilon}{3}t_m + \tfrac{\epsilon}{3}t_m + \tfrac{\epsilon}{3} \\
&\leq 2t_m\tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} \\
&\leq \tfrac{2\epsilon}{3} + \tfrac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Therefore, Σ is a uniformly continuous. Thus, Σ is a $(C(I), I)$ multi-selection for I.$\Box$

This result, together with the following lemma, gives the required map.

Lemma 3.1 (Domain Transfer) *Let $\mathcal{H} \subset 2^X$, and let A and B be compacta such that there exists a continuous surjection $k : B \to A$. If there exists a $(\mathcal{H}, A)$ multi-selection for X, then there exists a $(\mathcal{H}, B)$ multi-selection for X as well.*

PROOF OF LEMMA:

Let $\Sigma : \mathcal{H} \to X^A$ be a multi-selection, and let $k : B \to A$ be an arbitrary continuous surjection. We define the map $\Sigma' : \mathcal{H} \to X^B$ pointwise as follows: For each $S \in \mathcal{H}$, let $\Sigma'(S) = \Sigma(S) \circ k : B \to X$. Then Σ' is trivially a $(\mathcal{H}, B)$ multi-selection for X.$\Box$

3.1 $(C(X), C_{1/3})$ Multi-Selectible Continua

A question that remains at this point is whether or not there is a complete characterization of those continua which posses the property of "multi-selectibility." What follows is such a characterization. We will state the theorem now, and then we will proceed prove the theorem in the next several sections.

Theorem 3.2 (Main) *There exists a $(C(D), C_{1/3})$ multi-selection for D if, and only if, D is a dendrite.*

Definition 3.1 *Let D be a dendroid. For A, B disjoint subcontinua of D, and $x \in D \backslash A$, and $y \in D$, define*

1. *the juxtaposition*

$$xy = \begin{cases} \text{the unique arc irreducible between points } x \text{ and } y, & x \neq y \\ x, & x = y \end{cases}$$

2. *$xA = Ax =$ the unique irreducible arc between x and set A*

3. $AB =$ *the unique arc irreducible between the sets* A *and* B

4. $\hat{A}B = AB \cup A$

5. $A\hat{B} = AB \cup B$

6. $\hat{A}\hat{B} = AB \cup A \cup B$

If $E, F \in C(D)$ *and* $E \cap F \neq \emptyset$, *we will define*

7. $EF = \hat{E}F = E\hat{F} = \hat{E}\hat{F} = \emptyset$

For a set $S \subset D$, $\overline{S}$ *will denote the closure of* S *in* D. *We let* $\mathcal{N}$ *denote the set of positive integers.*

3.1.1 Proof of Sufficiency

Let D be a dendrite. We will prove that there exists a $(C(D), C_{1/3})$ multi-selection for D by construction:

Definition 3.2 ("First-Point Map") *Let* A *be a proper subcontinuum of dendrite* D, *and let* $x \in D - A$. *Then there exists a unique point* $a \in A$ *such that* $xA = xa$. *[That is,* $\{a\} = xA \cap A$ *for some* $a \in A$.*] Thus we may define the "first-point map for* A*"* $f_A : D - A \to A$ *by assigning to each* $x \in D - A$ *its appropriate point* $a \in A$ *as described.*

Definition 3.3 ("First-Point Retraction") *Next we define the "first-point retraction to A"* $r_A : D \to A$ *as follows: For each* $x \in D$,

$$r_A(x) = \begin{cases} x, & x \in A \\ \\ f_A(x), & x \in D - A \end{cases}$$

Both the first-point map for A and the first-point retraction to A are known to be continuous when D is a dendrite [7, Lemma 10.25].

Let $g : I \to D$ be a fixed arbitrary continuous surjection — guaranteed to exist since D is a locally connected continuum. We now define a function Σ_g which we will prove to be a $(C(D), I)$ multi-selection for D. Once this is established, we will employ the domain transfer lemma to demonstrate a $(C(D), C_{1/3})$ multi-selection for D as well.

Definition 3.4 (Function Σ_g) *Let function* $\Sigma_g : C(D) \to D^I$ *be defined pointwise as follows: For each* $A \in C(D)$, *let*

$$\Sigma_g(A) = r_A \circ g : I \to D$$

Since D is a dendrite, our definition of Σ_g is well-defined. It is obvious as well that $\Psi \circ \Sigma_g$ is the identity on $C(D)$. To prove the continuity of Σ_g, we will show that, for any sequence $\{B_n\}_{n=1}^{\infty}$ with $A \in C(D)$, $B_n \in C(D)$ for all $n \in \mathcal{N}$, and $B_n \to A$, the sequence $\Sigma_g(B_n) \to \Sigma_g(A)$. To do so, however, will require a few steps establishing the behavior of the sequence $\{B_n\}_{n=1}^{\infty}$ as it converges to A.

So, for the remainder of the proof, let $\{B_n\}_{n=1}^{\infty}$ be an arbitrary convergent sequence of subcontinua of D with $\lim_{n\to\infty} B_n = A$.

Our first proposition will be a corollary to the following theorem:

Theorem 3.3 (Goodykoontz [3]) *Let X be a continuum. Then the function*

$$f : 2^X \longrightarrow C(X)$$

given by

$$A \longmapsto \bigcap\{M \in C(X) \mid A \subset M\}$$

is continuous if and only if X is hereditarily unicoherent.

Proposition 3.1 ("Pair-wise Convergence") *The sequence $\{\hat{A}\hat{B}_n\}_{n=1}^{\infty}$ converges to A.*

Proof of Proposition:

Define sequence $S_n = B_n \cup A \in 2^X$. Clearly, $S_n \to A$. Since D is a hereditarily unicoherent continuum, it follows immediately from the theorem of Goodykoontz that $\hat{A}\hat{B_n} = f(S_n) \to f(A) = A$.$\Box$

With this knowledge, we may now consider the continua $\hat{A}\hat{B}_n$ as a whole, allowing us the following particular implementation of "zero-regular convergence."

Proposition 3.2 (Nearness of Retraction) *Let A and $\{B_n\}_{n=1}^{\infty}$ be as above. Then for any $\epsilon > 0$, there exists an $N \in \mathcal{N}$ such that, whenever $n > N$, the following properties are satisfied:*

1. $\forall x \in B_n,\ d(x, r_A(x)) < \epsilon$

2. $\forall x \in A,\ d(x, r_{B_n}(x)) < \epsilon$

Proof of Proposition:

Let $\epsilon > 0$ be given. By the pair-wise convergence proposition, we know that the sequence $\{\hat{A}\hat{B}_n\}_{n=1}^{\infty}$ converges to A. Since D is a dendrite, we have that $\hat{A}\hat{B}_n \to A$ zero-regularly; thus, there exists a $\delta > 0$ and an $N_1 \in \mathcal{N}$ such that, whenever $n > N_1$ and whenever $b \in B_n$ and $a \in A$ such that $d(a, b) < \delta$, we have that $\text{diam}(ab) < \epsilon$. Also, since $B_n \to A$ with respect to the Hausdorff metric, there exists an $N_2 \in \mathcal{N}$ such that, for all $n > N_2$, $H_d(A, B_n) < \delta$.

Let $N = \max\{N_1, N_2\}$, and let $n > N$. Then since $n > N_2$, for each $b \in B_n$, there exists an element $a_b \in A$ such that $d(b, a_b) < \delta$. Since $n > N_1$, we have that the points b and a_b lie in a connected set of diameter less than ϵ. Then $b\, a_b$ is a subset of this connected set; hence, $\text{diam}(b\, a_b) < \epsilon$.

Now, if $b \in A$, we have (by definition) that $d(b, r_A(b)) = 0 < \epsilon$. So consider the case when $b \notin A$. Then the point $r_A(b) \in A$ is necessarily an element of $b\, a_b$, and thus the arc $b\, r_A(b) \subset b\, a_b$ also has diameter less than ϵ. Hence, $d(b, r_A(b)) < \epsilon$.

A symmetric argument shows that for the same value N, we necessarily have that $d(a, r_{B_n}(a)) < \epsilon$.$\square$

The last piece of information guarantees us that, for any $\epsilon > 0$, we may find an $N \in \mathcal{N}$ such that the motion of point retractions from B_n to A and *vice versa* have distances bounded by ϵ. The real importance, however, is demonstrated in the following proposition, as these values are precisely what we need to bound to ensure the convergence of our functions $\{\Sigma_g(B_n)\}_{n=1}^{\infty}$.

Proposition 3.3 (Σ-Distance) *Let $n \in \mathcal{N}$. Then*

$$\rho(\,\Sigma_g(A), \Sigma_g(B_n)\,) \;\leq\; \max_{x\in A}\, d(x, r_{B_n}(x)) + \max_{x\in B_n}\, d(x, r_A(x))$$

Proof of Proposition:

We begin with the following observation: Let $S, T \in C(D)$ such that $S \subset T$. Then $\Sigma_g(S) = (r_S|_T \circ \Sigma_g(T))$. Therefore, $\rho(\Sigma_g(S), \Sigma_g(T)) = \max_{t\in T}\, d(t, r_S(t))$.

So let $C = \hat{A}\hat{B}_n$. Then by the "triangle inequality," we know that

$$\begin{aligned} \rho(\Sigma_g(A), \Sigma_g(B_n)) \quad &\leq \rho(\Sigma_g(A), \Sigma_g(C)) + \rho(\Sigma_g(C), \Sigma_g(B_n)) \\ &\leq \max_{x\in C}\, d(x, r_A(x)) + \max_{x\in C}\, d(x, r_{B_n}(x)) \end{aligned}$$

More specifically, however, note that points of $AB_n - (A \cup B_n)$ (assuming $AB_n \neq \emptyset$) do not concern us, since if A and B_n are disjoint then it is clear first that $AB_n = a\,b_n$ for some $a \in A$ and some $b_n \in B_n$ (*i.e.,* points a and b are the endpoints of the (unique) irreducible arc between A and B). Next, then, note that under our retractions, $r_A(B_n) = \{a\}$ and $r_{B_n}(A) = \{b_n\}$. Therefore,

$$\begin{aligned}
\rho(\Sigma_g(A), \Sigma_g(B_n)) &= \max\{\ \max_{x\in A}\ d(x, b_n),\ \max_{x\in B_n}\ d(a, x)\ \} \\
&= \max\{\ \max_{x\in A}\ d(x, r_{B_n}(x)),\ \max_{x\in B_n}\ d(x, r_A(x))\ \} \\
&\leq \max_{x\in A}\ d(x, r_{B_n}(x)) + \max_{x\in B_n}\ d(x, r_A(x))
\end{aligned}$$

Thus, the points of $AB_n - (A \cup B_n)$ are inconsequential in determining the distance between the functions, and we may revise our last inequality to read:

$$\rho(\Sigma_g(A), \Sigma_g(B_n)) \leq \max_{x\in B_n - A} d(x, r_A(x)) + \max_{x\in A - B_n} d(x, r_{B_n}(x))$$

Keeping in mind that $r_A(x) = x$ when $x \in A$ and that $r_{B_n}(x) = x$ when $x \in B_n$, we conclude that

$$\rho(\ \Sigma_g(A), \Sigma_g(B_n)\ )\ \leq\ \max_{x\in B_n} d(x, r_A(x)) + \max_{x\in A} d(x, r_{B_n}(x))\Box$$

We now have all of the tools to show that Σ_g is a $(C(D), I)$ multi-selection for D:

Let $\epsilon > 0$ be given. By the nearness of retractions proposition, there exists an $N \in \mathcal{N}$ such that whenever $n > N$,

1. for each $b \in B_n$, $d(b, r_A(b)) < \epsilon/2$, and

2. for each $a \in A$, $d(a, r_{B_n}(a)) < \epsilon/2$.

Observe then that, by the Σ-distance proposition we have that

$$\begin{aligned}
\rho(\Sigma(A), \Sigma(B)) &\leq \max_{b\in B} d(b, r_A(b)) + \max_{a\in A} d(a, r_B(a)) \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

Thus, the sequence $\{\Sigma(B_n)\}_{n=1}^{\infty}$ converges (uniformly) to $\Sigma(A)$. This proves the continuity of Σ.

It is clear that for two distinct elements $A, B \in C(D)$ we have that $\Sigma(A) \neq \Sigma(B)$ (following from the fact that $r_A \neq r_B$) — thus Σ is injective. Further, it is also a trivial matter that $\Psi \circ \Sigma = i_{C(D)}$. Therefore, Σ is a $(C(D), I)$ multi-selection for D.

Lastly, letting $k : C_{1/3} \to I$ be an arbitrary continuous surjection, application of the domain transfer lemma gives us the required $(C(D), C_{1/3})$ multi-selection for D.□

Corollary 3.1 ("(K, p) Choice") *Let D be a dendrite, K a subcontinuum of D, and p an element of K. Then there exists a continuous selection $\sigma_{(K,p)} : C(D) \to D$ such that $\sigma_{(K,p)}(K) = p$.*

Proof of Corollary:

By the sufficiency theorem, we know there exists a $(C(D), I)$ multi-selection for D. The remainder follows from the evaluation theorem.□

Corollary 3.2 *If D is a non-degenerate dendrite, then there exist uncountably many continuous selections of the form $\sigma : C(D) \to D$.*

Proof of Corollary:

Since D is a dendrite, we know there do exist $(C(D), I)$ multi-selections for D. By the (K, p) choice corollary, we know that for each $p \in D$, there exists a selection $\sigma_p : C(D) \to D$ such that $\sigma_p(D) = p$. Since D is non-degenerate, we have uncountably many distinct selections.□

3.1.2 Proof of Necessity

In order to show that the existence of a $(C(D), C_{1/3})$ multi-selection for D implies that D is a dendrite, we will first prove the following theorem which will stand alone. Our result will follow as a corollary.

Before reading the theorem, though, recall that for the $(C(D), C_{1/3})$ multi-selection to exist, we must first have that D is selectible (*i.e.,* there exists a continuous selection $\sigma : C(D) \to D$). By [8], we know that if a continuum X admits a selection for $C(X)$, then X is a dendroid, though it should also be noted now that this is not a sufficient condition. While it is known that certain classes of dendroids do admit selections (*c.f.,* [6, (5.8) and (5.9)], and [2, section 8]), sufficient conditions continue to elude researchers [6, Question (5.11)].

So, when searching for our conditions for the existence of multi-selections, we will begin with nothing less than a selectible dendroid.

Theorem 3.4 (Fixed Selection Theorem:) *Let D be a selectible dendroid. If D is not locally connected, then there exists a non-degenerate subcontinuum K of D and a point $p \in K$ such that, for* any *continuous selection $\sigma : C(D) \to D$, $\sigma(K) = p$.*

Proof of Theorem:

Let D be a selectible dendroid which is not locally connected. Then, there exists a continuum of convergence A, an open set $V \supset A$, and a sequence $\{A_n\}_{n=1}^{\infty}$ of subcontinua of D such that the components in V containing the A_n are mutually disjoint, and $A_n \to A$.

There exists a sub-sequence $\{A_{n_m}\}_{m=1}^{\infty}$ such that $\hat{A}_{n_m} A \to K$, for some subcontinuum K of D which contains A. Let $A_m = A_{n_m}$, $m \in \mathcal{N}$.

Now define the set S in the following way:

$$S = \{x \in D \mid x \in A_m A, \text{ for all but a finite number of } n\}$$

Claim 3.1 *$\overline{S}$ is either an arc, a point, or empty.*

Proof of Claim:

Suppose S is non-empty and non-degenerate. Let a and b be elements of S, and suppose that $aA \not\subset bA$ and that $aA \not\supset bA$.

The conflict is clear: the arcs aA and bA are subsets of arcs of the form $A_m A$ for all but a finite number of $m \in \mathcal{N}$. Therefore, it cannot be the case that points a and b both are elements of $A_m A$ for all but a finite number of m, as $a \notin bA$ and $b \notin aA$.

Therefore, all points of S may be totally ordered by

$$a \leq b \Longleftrightarrow aA \subset bA$$

Since D is a dendroid (containing no simple closed curves), and since $\overline{S}$ is an a-triodic subcontinuum of D, $\overline{S}$ is necessarily an arc.$\square$

We now consider two mutually exclusive cases — namely, the case when $\overline{S}$ is an arc, and the case when $\overline{S}$ is not an arc. As the reader will shortly see, the arguments are slightly different, though the results are as claimed.

Case One: $\overline{S}$ is an arc.

Let p be the end-point of $\overline{S}$ which is not an element of A. Then for some sub-sequences $\{x_i\}_{i=1}^{\infty}$ and $\{A_{m_i}\}_{i=1}^{\infty}$ we have that

1. $x_i \in A_{m_i} S \cap S = \{x_i\}$

2. $x_i \to p$

3. $\hat{A}_{m_i}\, x_i \to K'$, for some subcontinuum K' of K

Define $A_i = A_{m_i},\ i = 1, 2, \ldots$

Claim 3.2

$$\overline{S} \subset K'$$

PROOF OF CLAIM:

Suppose otherwise.

Let $a \in pA \cap A = \{a\}$. Clearly both $p, a \in K'$ (with $p \neq a$, as they are opposite end-points of the same arc). So there exists some point $q \in \overline{S} - \{p, a\}$ and an open neighborhood U of q such that $U \cap A_i x_i = \emptyset$ for all but a finite number of $i \in \mathcal{N}$. Since K' is connected and has both p and a as elements, $pa \subset K' - U$. However, $q \in pa \subset \overline{S}$ as well, so this set has a non-empty intersection with U. Therefore, D contains a simple closed curve — contradicting the hereditary unicoherence of D.

Therefore, $\overline{S} \subset K'$.$\Box$

From the above, we now observe that the two sequences $\{\hat{A}_i\, A\}_{i=1}^{\infty}$ and $\{\hat{A}_i\, x_i\}_{i=1}^{\infty}$ both converge to K'. By definition, however, we know that every subsequence of $\{\hat{A}_m\, A\}_{m=1}^{\infty}$ converges to K; therefore, $\hat{A}_i\, A \to K' = K$.

In summary,

1. $\hat{A}_i\, S \to K$

2. $K \to K$ (a constant sequence)

3. $K \cap \hat{A}_i\, S = \{x_i\}$

4. $x_i \to p$

Thus, $\{p\}$ is a "bend set for K" as defined by Maćkowiac, and by his work we know that for any continuous selection $\sigma : C(D) \to D$, we must have that $\sigma(K) = p$ [4].

Case Two: $\overline{S}$ is not an arc.

Then S is either empty or a singleton. For each $i \in \mathcal{N}$, let $x_i \in \hat{A}_i\, A \cap A = \{x_i\}$. There exists a sub-sequence $\{x_{i_j}\}_{j=1}^{\infty}$ such that $x_{i_j} \to p$ for some $p \in A$. [Define $x_j = x_{i_j}$ and $A_j = A_{i_j}$.]

The sequence $\{\hat{A}_j\, A\}_{j=1}^{\infty}$ remains convergent to K. Further, note that since K necessarily intersects the boundary of V and the set A does not, set K contains A. Therefore we have the following:

1. $\hat{A}_j\, A \to K$

2. $K \to K$ (a constant sequence)

3. $\{x_j\} = A_j A \cap A$

4. $x_j \to p$

Therefore, $\{p\}$ is a bend set for K; hence, for any continuous selection $\sigma : C(D) \to D$, we must have $\sigma(K) = p$.$\square$

We see that, in either case, we have found a non-degenerate subcontinuum K of D and a point $p \in K$ such that, under any continuous selection $\sigma : C(D) \to D$, $\sigma(K) = p$. This completes the proof of the theorem.$\square$

Corollary 3.3 (Necessity) *Let D be a selectible dendroid and let A be a compactum. If there exists a $(C(D), A)$ multi-selection for D, then D is in fact a dendrite.*

Proof of Necessity:

Let D be a selectible dendroid such that there exists a $(C(D), A)$ multi-selection for D. If D is a singleton set, then of course D is a dendrite. So suppose that D is non-degenerate. Let K be a non-degenerate subcontinuum of D. Then by the (K, p) choice corollary, for every point $p \in K$, there exists a continuous selection $\sigma_{(K,p)} : C(D) \to D$ such that $\sigma_{(K,p)}(K) = p$. Since both K and p were arbitrary, it follows from the contrapositive form of the fixed selection theorem that D is locally connected and, thus, D is a dendrite.□

4 "Pinning Down" Dendrite Selections

We now give an application of the multi-selection existence theorem and the (K, p) choice corollary:

Theorem 4.1 (Finite Choice) *Let D be a dendrite with metric d, and let $\mathcal{F}$ be a non-empty finite family of ordered pairs defined*

$$\mathcal{F} = \{(K_i, p_i) \mid K_i \in C(D), p_i \in K_i, i = 1, 2, \ldots n\}$$

*such that the K_i are distinct (*i.e., $K_i = K_j \Longrightarrow i = j$*). Then there exists a continuous selection $\sigma_{\mathcal{F}} : C(D) \to D$ such that, for $i = 1, 2, \ldots, n$,*

$$\sigma_{\mathcal{F}}(K_i) = p_i, \; i = 1, 2, \ldots, n$$

PROOF OF THEOREM:

Let $g : I \to D$ be a continuous surjection. Then, by the (K, p) choice corollary, for each $i = 1, 2, \ldots, n$, there exists an $x_i \in I$ such that

$$[\Sigma_g(K_i)](x_i) = p_i$$

Further, since our family $\mathcal{F}$ is finite and the K_i are distinct, there exists an $\epsilon > 0$ such that, for each i,

$$\mathcal{N}(\epsilon, K_i) \bigcap (\bigcup_{j \neq i} \mathcal{N}(\epsilon, K_j)) = \emptyset$$

Define function $\phi : C(D) \to I$ pointwise as follows: For each $A \in C(D)$, let

$$\phi(A) = \begin{cases} 1, & H_d(A, K_i) \geq \epsilon, \forall i \\ \frac{H_d(A, K_c)}{\epsilon}(1 - x_c) + x_c, & \exists\, c \text{ such that } A \in \mathcal{N}(\epsilon, K_c) \end{cases}$$

The function ϕ is well-defined, as the ϵ-neighborhoods about the various K_i are disjoint. Inside each such neighborhood, ϕ is linear; outside all such neighborhoods, ϕ is constant (valued 1). Lastly, ϕ continuous at points of the neighborhood boundaries. Therefore, ϕ is continuous. Lastly, observe that $\phi(K_i) = x_i, \; i = 1, 2, \ldots, n.$

Now define function $\sigma_{\mathcal{F}} : C(D) \to D$ pointwise as follows: For each $A \in C(D)$, let

$$\sigma_{\mathcal{F}}(A) = [\Sigma(A)](\phi(A))$$

Since Σ is continuous, since the evaluation of Σ is continuous, and since function ϕ is continuous, we have that function $\sigma_{\mathcal{F}}$ is continuous. Lastly, since $\phi(K_i) = x_i$ and $[\Sigma(K_i)](x_i) = p_i$, we have $\sigma_{\mathcal{F}}(K_i) = p_i$ for each $i = 1, 2, \ldots, n$ as desired.$\square$

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