Implementing the Empirical Stone Mine Pillar Strength Equation into the Boundary Element Method Software LaModel

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Implementing the Empirical Stone Mine Pillar Strength Equation into the Boundary Element Method Software LaModel

Samuel Escobar

Thesis submitted to the Benjamin M. Statler College of Engineering and Mineral Resources at West Virginia University in partial fulfillment of the requirements for the degree of Master of Science in Mining Engineering

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2021

Keywords: Limestone Pillars, Pillar Strength, Boundary Element Method Software, S-Pillar, LaModel.

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Implementing the Empirical Stone Mine Pillar Strength Equation into the Boundary Element Method Software LaModel

Escobar, Samuel

Underground stone mines in the United States (U.S.) generally use the room-and-pillar mining method, and designing stable pillars and knowing their behavior through the mine's life is necessary to avoid ground control hazards. In the U.S., empirical pillar strength equations and the S-Pillar program were developed to assist in the design of stable pillars for room-and-pillar workings in underground stone mines (Esterhuizen et al., 2011).

S-Pillar software assesses the stability of stone mine pillars by calculating the factor of safety and comparing it with the historical data on pillar performances. The factor of safety is the ratio of the pillar strength, calculated using the empirical strength equation (Esterhuizen et al., 2011), to the pillar stress. S-Pillar uses the tributary area method to calculate the stress on the pillars, and this method assumes that the full weight of the overburden is uniformly distributed among pillars. Moreover, S-Pillar calculates the pillar load as the maximum depth over the pillar layout, and the stress calculation is only truly valid if the areas of the mine use regular-sized pillars (Esterhuizen et al., 2011).

The S-Pillar approach is an empirical coal pillar stability program. Heasley (1998) developed the LaModel software, a Displacement Discontinuity variation of boundary element method, and used concentric rings of increasing strength coal materials to numerically simulate the empirical coal strength equation in LaModel. Heasley et al. (2010) indicated that the integration of empirical coal strength equation in LaModel allows empirical pillar stability analysis to be extended over
complex mine geometries and variable topography. Essentially, LaModel can often simulate an empirical strength equation with concentric material bands, and this study aims to expand the usefulness of the empirical stone mine pillar strength equation (and S-Pillar software) by integrating it into LaModel to allow stress and factor of safety analysis with complex geometries and variable topography.

In this research, the equation for the increasing stress into the interior of a stone mine pillar, as a function of the pillar width-to-height ratio, is derived from the empirical pillar strength equation proposed by Esterhuizen et al. (2011). The gradient stress equations for the stone mine pillars were derived by following similar approaches to those presented by Mark et al. (1992) and Johnson et al. (2014). In these approaches, it is assumed that the variation of stresses within the pillar is a function of distance to the closest rib. The stress gradient function provides the stress distribution within the pillar and is used to derive concentric rings of zones to simulate stone mine pillar yielding in boundary element software. This stress gradient function assumes that there are no large discontinuities present and that discontinuities do not have an impact on the strength calculation. The pillar geometry and rock mass parameters for testing the stress functions are selected from the S-Pillar database. The functions are tested for pillars with different width-to-height ratios and different element sizes. Finally, a beta version of the “stone pillar wizard,” which will be implemented into LaModel’s preprocessor, Lampre, is developed using the derived equations to obtain the Stone Pillar material properties. Using the new stone pillar wizard in Lamodel, the pillar stress distribution, pillar safety factors, overburden stress, and all the other output stress items available in the LaModel software can be determined for an underground limestone mine.

Keywords Limestone Pillars, Pillar Strength, Boundary Element Method Software, S-Pillar, LaModel.
To My Mother Amparo,

My Father Ivan,

My Brother Juan Jose,

and,

My Grandparents Marta, Nestor, Omaira, and Antonio.
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Chapter 1 Introduction

Limestone is a rock composed mostly of calcium carbonate (CaCO₃) and, although it is not as widely known as a significant contributor to the economy as precious metals (gold, silver, or platinum) or base metals (copper, lead, or zinc), it is an essential indicator of the economic development of a nation (Shinobe et al., 1997). Limestone is used in the construction, environmental and agricultural industries, being its primary use in the production of Portland cement. In addition, it is used for asphalt production and as an aggregate in concrete (USGS, 2008). Generally, limestone mining operations have been a major source of mining employment close to towns and have produced a common raw material for the development of cities.

Traditionally, surface mining methods have been applied to extract limestone (Herrera et al., 2006). In 2020, approximately 97% of stone mining operations in the U.S. were surface mines (NIOSH, 2021). Moreover, according to the Mineral Commodity Summaries of 2021, the surface production of crushed stone in the U.S. accounted for approximately 1.41 billion tons. Nevertheless, surface limestone mining constantly struggles to comply with environmental regulations because of its close location to urban areas. Shinobe et al. (1997) stated that dust control, noise, vibrations, or visual impact are the most common issues. Therefore, obtaining permits to develop or operate surface limestone mines has become increasingly difficult. Furthermore, the limestone deposits are significantly deeper, and their extraction costs are becoming much higher (Parker, 1996). Therefore, underground limestone mining has become an essential solution for the limestone industry. Parker (1996) stated that the transition to underground limestone mining would bring benefits such as minimizing communities concerns or access to deposits with a higher grade. However, in underground mining operations, a different group of issues must be considered to assure a safe working environment. Iannacchione (1999) stated that as limestone mines transition into underground workings, it is essential to have adequate stone pillar design methods.

1.1. Problem Statement

Underground stone mining represents around 21% of the total underground mining operations in the United States (NIOSH, 2021). In 2019 more than 2000 people worked in underground stone mining (NIOSH, 2019). Hence it is essential to ensure safe working conditions for all those who
are involved in the industry. Approximately 40% of fatalities have been linked with falls of ground from roofs and pillars in underground stone mines since 2006, and the time lost related to ground control issues represents about 15% of the total lost working days in underground stone mining (MSHA, 2016). In addition, Esterhuizen et al. (2011) stated that unstable pillars might mobilize roof/rib falls and pillar collapses. Therefore, it is vital to design stable pillars and know their behavior throughout the mine's life.

In the U.S., the S-Pillar program was developed by the National Institute for Occupational Safety and Health (NIOSH) to assist in designing stable pillars in underground stone mines (Esterhuizen et al., 2011). The S-Pillar program is used to calculate the safety factor of the limestone pillars and compare this safety factor with the historical data to guide mine engineers to select suitable pillar size to ensure the local and global stability of the mine. In addition, S-pillar also considers the influence of a large joint set intersecting a pillar on the stability of the pillar. This approach does not consider the relative location of geological structures with respect to pillars or that multiple joint sets may intersecting a pillar. Moreover, the S-Pillar program conservatively calculates the pillar load as the maximum depth over the pillar layout, and the tributary-area stress calculation is only truly valid if the mine use regular-sized pillars (Esterhuizen et al., 2011). Currently, the S-Pillar program calculates the factor of safety of underground stone mines by assuming that the full weight of the overburden is evenly distributed among the pillars and is only valid if large areas are mined using regular-size pillars (Esterhuizen et al., 2011). Therefore, the S-Pillar program can be further improved by extending the stability analysis of pillar systems to variable topographies and geometries.

1.2. Objective of Thesis

This research aims to improve the safety of underground limestone mines by extending the application of the S-Pillar stone pillar strength equation to Boundary Element Method software (BEM). Once implemented into a BEM program stone mine operators will be able to more accurately calculate pillar safety factors due to variable topography and with variable pillar sizes. The main hurdle to implementing the stone pillar equation into a BEM program will be the derivation of a gradient stress equation for stone pillars from the empirical pillar strength equation proposed by Esterhuizen et al. (2011). This gradient stress equation provides the stress distribution within the pillar, and it is used to derive concentric rings of material to simulate stone mine pillar...
yielding in the boundary element method software LaModel (Heasley, 1998) that is widely used in coal mining to analyze the stability and stress conditions of pillars.

1.3. Statement of Work

In this thesis, the gradient stress equation for stone mine pillars, function of pillar width-to-height ratio, is derived from the empirical base pillar strength equation proposed by Esterhuizen et al. (2011). In particular, the gradient stress equation for stone mines is derived by following a similar approach to that presented by Mark et al. (1992) and Johnson et al. (2014), where it is assumed that the variation of strength within the pillar is a function of the shortest distance to the rib boundary. Following the approach used by Heasley (1998) for coal pillars in LaModel, the stone pillar stress-strain relationships are derived for different cell elements subject to confinement. The two most common cell elements used to model pillars in boundary element programs are the rib cell element and corner cell element (Johnson et al., 2014), therefore in this study, the gradient stress equation for stone pillars is obtained for rib cell and corner cell elements. Additionally, this study considers the LDF factor (Esterhuizen et al., 2011) to be 1.0. Therefore, it is assumed that there are no large discontinuities present and that the pillar’s strength is not affected by them.

Once the cell pillar strength functions were obtained, they were tested against the S-Pillar equation (Esterhuizen et al., 2011). Pillar dimensions and rock mass parameters for verifying the equations, relative to S-Pillar strength estimates, were selected from the S-Pillar database (Esterhuizen et al., 2011). Then, the equations were tested for pillars with different width-to-height ratios. Once the variable strength equations have been tested for different width-to-height ratios, the behavior of the equation for different element sizes is also verified. Finally, the variable strength equations implemented into LaModel are used to assess the stability of an underground limestone mine operations in the Loyalhanna formation.

This study consisted of four specific tasks: 1) Derivation of the stress gradient equations for stone mines; 2) Test the stress gradient equation against the empirical stone pillar strength equation (Esterhuizen et al., 2011); 3) Implement the stress gradient equations into LaModel; 4) Use the new method in a case study mine. A detailed explanation of these tasks is shown below:

- **Task 1:** Derivation of the stress gradient equations for the rib cell and corner cell elements for stone pillars.
Task 2: Test and calibrate the stress gradient equations against the empirical base pillar strength equation proposed by Esterhuizen et al. (2011) using the stone mine parameters from the S-Pillar database (Esterhuizen et al. 2011).

Task 3: Once the stress gradient equations are calibrated and present optimum results, a beta version of the stone pillar wizard, will be implemented into LaModel's preprocessor Lampre.

Task 4: Use the new method on a stone mine case study.

1.4. Thesis Outline

This thesis is organized into five chapters. The chapters are described as:

Chapter 1 The introduction chapter provides a brief introduction to the topic, the problem statement, the objective of the thesis, and the statement of work.

Chapter 2 Reviews the literature by discussing the studies on stone pillar strength and stress gradient equations for different pillars.

Chapter 3 Explains the methodology used to obtain and test the cell strength equations for stone mine pillars. Moreover, it presents the derivation of the post-peak behavior for the cell strength equations.

Chapter 4 Use the cell strength equations as input for LaModel software to obtain the stress output from LaModel for a case study mine in the Loyalhanna Formation.

Chapter 5 Provides the conclusions and future recommendations for this research.
Chapter 2 Literature Review

2.1. Introduction

This chapter presents: (i) a short summary of empirical pillar design methods in underground coal, hard rock, and specifically stone mining and (ii) observed pillar performance and failure mechanisms at underground stone mining in the United States (U.S.). In addition, the empirical strength equation for underground stone pillars (Esterhuizen et al., 2011) developed by NIOSH and the strength and pillar cell equations that have been developed for cell elements in coal pillars used in the boundary element codes are discussed in detail since the research methodology used in this thesis was influenced from these studies.

2.2. Underground Pillar Design Considerations

Brady et al. (1985) stated that in underground mining methods where pillars are used as the primary support for the excavations, pillar system should be designed to control rock mass displacements in the mine near-field domain (Figure 2.1). Underground limestone mines in the United States are typically operated using the room-and-pillar mining method. These mines are located in relatively flat-lying deposits in the Eastern and Midwestern U.S. In this type of mining, pillars serve as the overburden support, providing a safe work environment for everyone involved with the underground workings (Esterhuizen et al., 2008).

Figure 2.1. Schematic illustration of mine near-field stability affected by different aspects of mine design (Brady et al., 1985).

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Brady et al. (1985; 2004) defined pillar strength as the maximum resistance of the pillar to axial compression, stating that it is closely related to the pillar shape and volume. Esterhuizen et al. (2008) stated that the average pillar stress in a flat-lying orebody could be estimated by the tributary area method where the overburden weight is homogeneously distributed among all the pillars. Esterhuizen et al. (2008) indicated that pillars not only provide global stability, but also support the immediate roof strata, and that roof collapses in underground can result from unstable pillars. Furthermore, Esterhuizen et al. identified that the key factors necessary to design stable stone pillars are to correctly estimating the pillar stress and strength used to obtain the pillars' factor of safety (FOS).

\[
FOS = \frac{\text{Average Pillar Strength} (S)}{\text{Average Pillar Stress} (\sigma_p)}
\]  

Brady et al. (1985; 2004) defined pillar failure as when the pillar is subject to a stress higher than its peak resistance. Moreover, he indicated that the pillar's response to load is effected by the rock mass geomechanical properties, existing geological structures in the rock mass, pillar geometry, and dimensions. Design an appropriate pillar factor of safety is a crucial aspect for safety in a room and pillar mine layout. Its selection requires consideration of several aspects, such as variable geology, overburden, and mining-induced stress distributions. Empirical design methodologies define design safety factor based on the analysis of historical failed and successful cases of pillar stability and on-site pillar performance evaluations (Esterhuizen et al., 2008).

In 1997, Lunder and Pakalnis indicated that a pillar's failure mobilizes by progressive stages of degradation, as presented in Figure 2.2. The initial signs of rock stress are local shear failure, and the presence of rib spalling indicating fracture initiation and rock damage along the pillar. At this initial stage, the pillar is partially failed, yet the pillar's core has not failed. As the stress increases, damage accumulates through internal crack initiation, and eventually, when cracking is fully mobilized, the pillar is at its peak strength and a state of failure.
Figure 2.2. Schematic illustration of the evolution of failure in a pillar (Lunder and Pakalnis, 1997).

Mark et al. (1997) concluded that massive pillar collapses in underground mines follow a "domino-type" pattern. When a pillar fails, the load it carries is swiftly transferred to its neighbors, leading to large areas of a mine collapsing. Furthermore, the potential of failure in a pillar drastically increases as the width-to-height ratio decreases. The strength of slender pillar's decrease rapidly after it reaches its peak resistance (Salamon, 1970). In addition, Lane et al. (1999) observed that pillars stressed to their point of failure start developing an hourglass shape.

2.2.1. Underground Stone Pillar Performance

In 2011, Esterhuizen et al. presented a comprehensive study on design guidelines for stable pillars for underground limestone mines. In that study, the authors collected operational information, such as excavation dimensions, excavation stability, rock jointing, and rock mass classification from 34 underground limestone mines in the U.S. From the information collected, Esterhuizen et al. (2011) determined that from the 91 pillar layouts surveyed, just 18 individual pillars were considered failures. Furthermore, the authors stated that some pillars required ground support such as bolting for assuring local stability.
The 18 failed cases were assessed visually, and the modes of pillar instability were categorized as either crushing or structure-controlled failure (Esterhuizen et al., 2011). Crushing failure is defined as crushing and spalling of the rock with limited shearing along discontinuities. Rib spalling and the emergence of an hourglass shape is the indicator of crushing failure. This failure type is described as progressive rather than sudden, similar to the study by Lunder and Pakalnis (1997).
The structure-controlled failure mechanism is characterized by shearing along major discontinuities such as large joints or bedding planes. Failure by this mechanism occurs when sliding occurs along the discontinuity. Moreover, soft infill in the discontinuities can cause the adjacent rock to loosen or fracture, leading to progressive pillar failure.

2.2.2. Empirical Stone Pillar Strength Equation

Brady et al. (1985) defined pillar strength as the maximum resistance of the pillar to axial compression. Moreover, the authors stated that pillar strength is closely related to its shape and volume. Extensive research has been performed for the estimation of a pillar's strength over the years. A variety of empirical equations have been developed for estimating the strength of pillars. However, different strength formulae are calibrated for different locations, resulting in different pillar strengths (Peng, 2007). Recently, laboratory testing and numerical models analyses have contributed to a better understanding of pillar failure mechanisms and pillar strength (Esterhuizen et al., 2008).

Esterhuizen et al. (2011) developed an equation that estimates the strength of pillars in underground stone mines. A database of stone mine pillar performances was created by Esterhuizen et al. and from this database, the authors determined that the 18 individual cases of
stone mine pillars that failed were insufficient for developing an empirical strength equation. Therefore, Estehuizen et al. included results of numerical models and data of different types of hard rock mining operations.

Roberts et al. (2007) developed empirical pillar design guidelines in lead mines of the Viburnum Trend in Southeastern Missouri from back-analysis of pillar damage with DDA variation of boundary element model. Later, Esterhuizen et al. (2011) used this empirical information to derive the base strength equation used in S-Pillar. Esterhuizen et al. compiled information of stable and failed pillars from the lead mines in the Viburnum Trend as a reference point due to its flat-lying deposits and room and pillar mining operations. Furthermore, the authors evaluated the rock mass quality at several locations in the Viburnum and declared that they fell in the range of rock mass quality in stone mines (Esterhuizen et al., 2011). Roberts et al. (2007) simulated the strength of hard-rock pillars with different rib and core cell elements to simulate the effect of confinement on pillar strength. Following this approach, Esterhuizen et al. (2011) determined the pillar strength by subdividing the pillar into 8 ft square "inner" and "outer" elements, where the inner elements have a higher strength due to the confinement. In addition, Roberts et al. (2007) declared that the pillar's strength was dependent on the width of the pillar and was affected by its height. Therefore, in their methodology, Esterhuizen et al. (2011) considered the shape and volume of the pillar for the estimation of the pillar strength. Moreover, Esterhuizen et al. (2011) used this approach which follows the definition of pillar strength power equation developed by Brady et al. (1985; 2004).

In 2008, Esterhuizen et al. determined that a pillar strength equation that accurately incorporates the pillar shape and volume is a power equation in the form:

$$S = k \left( \frac{w^\alpha}{h^\beta} \right)$$  \hspace{1cm} (2.2)

The strength of the pillar is represented by $S$, $k$ is a rock strength parameter, $w$ is the width of the pillar, $h$ is the pillar's height, and $\alpha$ and $\beta$ are empirical parameters related to the geomechanical conditions of the rock mass (Esterhuizen et al., 2008).

Finally, Estehuizen et al. obtained an empirical base equation that estimates the strength of a stone pillar in the form of the power equation (equation 2.2) by developing a series of inner and outer element strength curves for different pillar widths and using the least squares curve fitting technique:
The parameter k can be expressed in terms of the Unconfined Compressive Strength (UCS) as:

\[ S = k \left( \frac{w^{0.3}}{h^{0.59}} \right) \]  (2.3)

The parameter k can be expressed in terms of the Unconfined Compressive Strength (UCS) as:

\[ k = 0.92 \times \text{UCS}, \text{ for the Imperial System of Units} \]  (2.4)
\[ k = 0.65 \times \text{UCS}, \text{ for the International System of Units} \]  (2.5)

Furthermore, Esterhuizen et al. developed a modification for the empirical base strength equation (equation 2.3) that accounts for large discontinuities (equation 2.7). The authors developed a Discontinuity Dip Factor (DDF) table that represents the strength reduction of a pillar by the dip of the structure and the width-to-height ratio of the pillar. Moreover, they determined a frequency factor (F.F.) table related to the frequency of large discontinuities present in a stone pillar.

\[ S = k \times LDF \left( \frac{w^{0.3}}{h^{0.59}} \right) \]  (2.6)
\[ LDF = 1 - DDF \times FF \]  (2.7)

### 2.2.3 Stone Pillar Floor Benching

As Esterhuizen et al. (2011) stated, benching the floor between pillars is common in underground stone mines, where the formation thickness surpasses the initial development mining height. The authors elaborated that pillar instability was common among benched pillars. In addition, several cases showed that non-benched pillars at the perimeter of benching areas presented increased loading, therefore knowing the response of the pillars during benching is important from a design standpoint (Esterhuizen et al., 2011). Using numerical analysis, the authors investigated impact of the load and strength changes caused by benching on pillar strength. These analyses were based on the assumption that rock failure initiates by a process of brittle spalling. Esterhuizen et al. (2011) estimated that the benched pillar's strength reduced around 16% from the development stage with a width-to-height ratio of 1.0 and around 37% strength decrease for pillars with a width-to-height ratio of 1.5. Esterhuizen et al. (2011) stated that development pillars at the edge of benched areas (perimeter pillars) experience an average stress increment of around 12%. Moreover, the authors determined that "perimeter pillars" (pillars that are at the perimeter of the benching area) present higher stress than partially benched pillars due to the reduced stiffness in partially benched pillars. The height increment in one of the sides of the pillar causes a load transfer to stiffer non-benched pillars in the surroundings (Esterhuizen et al., 2011).
2.2.4. Failure Mechanisms in Stone Mine Pillars

Esterhuizen et al. (2011) stated that the hard rock extracted in stone mines could be classified as brittle rock, which tends to swiftly lose strength after the peak load-bearing capacity of the rock has been reached. Furthermore, the authors stated that failure in brittle rock surrounding excavations tends to initiate by spalling that consists of slabs of rock formed parallel to the excavation surfaces. According to Stacey (1981), spalling is a process that occurs when confining stress is low and the rock splits in a direction parallel to the major compressive stress and forms slabs. Moreover, Esterhuizen et al. (2011) reported that stone mine pillars tend to be relatively slender. Slender pillars behave differently than wider pillars due to the lack of core confinement. While in wide pillars, the central core of the pillar is confined and this confinement results in an increment in the strength of the pillar. In slender pillars, the confinement is insignificant or may be absent, resulting in lower pillar strength (Esterhuizen et al., 2011). Furthermore, Esterhuizen et al. (2011) stated that pillars with a width-to-height ratio of 0.5 presented tensile failure mechanism in the form of axial splitting. Wide pillars present a combination of tensile and shear failure, where shear failure occurs at the pillar's core, and tensile failure governs the pillar behavior at the outer elements (Esterhuizen et al., 2011; Suner et al., 2021).

Iannachione (1999) performed numerical simulations using a 2-dimensional finite difference code. These calculations were performed under plane-strain conditions. The author assumed that individual elements in the model behave in an elastic-perfectly plastic behavior when performing these simulations. Moreover, Iannachione (1999) stated that the overall pillar presented strain-
softening and strain-hardening behavior. The model pillars were subjected to very slow loading of roof and floor to load the pillar through several distinct strength phases gradually (Figure 2.7).

![Diagram showing elastic-plastic model with failure patterns](image)

*Figure 2.7. Elastic-plastic model which produces progressive failure patterns (Iannachionne, 1999).*

During the early stage of loading, the pillar displays elastic characteristics where the pillar deformation is proportional to increases in the average principal stress levels within the pillar. During this phase, minor yielding at the pillar edges occurs. An hourglass-shaped elastic core is produced in the next phase from the progressive failure of the pillar's outer perimeter (Iannachionne, 1999). Iannachionne (1999) observed that the maximum pillar strength is achieved when the highest vertical stress in the elastic core is supported by the maximum horizontal confinement available in the pillar. In addition, the author determined that any additional loading would result in a rapid loss of strength beyond that point. Iannachionne (1999) stated that the zone of plastic yield extended throughout the pillar producing residual strength (Figure 2.7, Point D).

Furthermore, Suner (2021) presented how the failure evolves in the loading stages for pillars with width-to-height ratios of 0.5 and 1.0 using the two-dimensional Universal Distinct Element Code (UDEC). The author established the different loading stages with instantaneous snapshots of the stress-strain behavior of the pillars together with the maximum stress contours and the joint plane states. For pillars with a width-to-height ratio of 0.5, Suner (2021) stated that tensile failure dominated the pillar behavior through all the loading stages. The author reported that the pillar sustains the load beyond its peak strength, and tensile failure governs all the pillar elements at the
final loading stage. For a pillar with a width-to-height ratio of 1, Suner (2021) reported that shearing along the joint planes at the initial loading phases governs the pillar behavior. In the third and fourth stages, the outer elements experience tensile failure, extending towards the core. In the final loading stage, after reaching the pillar's peak strength, shear failure still governs the core elements, but the tensile failure dominates the overall pillar, including some core elements.

2.3. Coal Pillar Strength Equations

In 1992 Bieniawski stated that two different approaches essentially determined coal pillar design. The progressive failure approach elaborates on the nonuniform stress distribution in coal pillars, where failure initiates at the most crucial point and propagates gradually to ultimate failure. On the other hand, the ultimate strength approach declares that the load-bearing capacity becomes zero when the ultimate strength is exceeded. Furthermore, Bieniawski (1992) indicated that for coal pillar design, the pillar strength could be estimated by size or volume effect, the geometry of the pillar, and the properties of the material, while the pillar load could be estimated using the tributary area method.

Pillar Design for coal mines in the U.S. goes back to the early 20th Century (Mark, 1999), and it has evolved until it reached a "standard" methodology in the 1980s. Where according to Mark (1999), three main steps were stated, 1) Estimate the pillar load, 2) Estimate the pillar strength, and 3) Calculate the Factor of Safety. Furthermore, Mark (1999) elaborates on the definition of pillar strength by stating that coal pillar strength can be estimated from empirical observations. Furthermore, the author stated that 900 psi is established as the accepted “average” in situ stress of coal.

In 1968 Bieniawski, developed a pillar strength formula (equation 2.8) based on in situ tests involving 66 large coal specimens (up to 2 meters in width and height) in coal mines in South Africa (Bieniawski et al., 1975). This equation can be used for both room and pillar and longwall coal mining (Bieniawski, 1968).

\[
\sigma_p = \sigma_i \left( 0.64 + 0.36 \frac{W}{h} \right)
\]  
(2.8)
Figure 2.8. Strength data from large-scale in situ tests on coal pillars (Bieniwaski et al., 1975).

Similar to Bieniawski's empirical strength (1968), several coal pillar strength formulas were obtained for different coal seams worldwide. Bieniawski (1968) states that there are mainly four other empirical formulas used to estimate the strength of coal pillars:

\[ \sigma_p = k \left( \frac{w}{h} \right)^{0.5} \]  \hspace{1cm} \text{Holland-Gaddy, 1964} \hspace{1cm} (2.9)

\[ \sigma_p = \sigma_i \left( 0.778 + 0.222 \frac{w}{h} \right) \]  \hspace{1cm} \text{Obert-Duvall, 1967} \hspace{1cm} (2.10)

\[ \sigma_p = 1,320 \left( \frac{w}{h} \right)^{0.46} \]  \hspace{1cm} \text{Salamon-Munro, 1967} \hspace{1cm} (2.11)

\[ \sigma_p = \sigma_i \left( \frac{w}{h} \right)^{0.5} \]  \hspace{1cm} \text{Holland, 1973} \hspace{1cm} (2.12)

Where \( \sigma_p \) is pillar strength, \( \sigma_i \) is the in situ coal strength, \( w \) is the width of the pillar, and \( h \) is its height.

Mark and Iannacchione (1992) compared 10 empirical strength formulas for coal pillars. They stated that some empirical strength formulas predicted an exponential increase, while others would increase until reaching a maximum limiting value. Some equations like the one proposed by Bieniawski (1968) showed a linear behavior in the strength increment as the width-to-height ratio increased.
2.3.1. Coal Pillar Variable Stress Equations

As stated by Mark and Iannacchione (1992), empirical formulas estimate the strength of a coal pillar by evaluating the pillar as a unique element. They are used to estimate the average strength of the pillar. Nevertheless, Wagner (1974) elaborated that the stress within the pillar is nonuniform at the ultimate load, and pillar failure is progressive and not sudden. Therefore, while empirical strength formulas are valuable for practical design, they don't show that some sections of the pillar yield before it is subject to its maximum load (Iannachionne et al., 1992).
Wilson (1973) proposed a "mechanics-based" approach to coal pillar design where the coal pillar is defined as constituted by a *yield zone* in the outermost part of the pillar that constraints a *confined core* in the innermost part of the pillar. In this study, the author declares that once the coal has yielded, there is no further loss of strength in the pillar. Furthermore, Wilson (1973) derived an expression for the vertical stress gradient within the yield zone (equation 2.13) by assuming coal follows a linear failure criterion such as Mohr-Coulomb:

\[
\sigma_v = p' \times k \times \left( \frac{x + \frac{h}{2}}{\frac{h}{2}} \right)^{k-1}
\]  

(2.13)

Where "\( \sigma_v \)" is pillar stress, \( p' \) is the unconfined compressive strength of the failed coal at the edge of the pillar, "x" is the distance from the rib to the center of the pillar, and "k" is the triaxial stress factor in terms of the internal angle of friction.

Barron (1984) presented a different approach to the one proposed by Wilson (1973), where the author stated that the exponential stress increment in the yield zone should reach a limiting value as the pillar stress increments at a decreasing rate. Moreover, Mark and Iannacchione (1992) stated that closed-form analytical solutions for estimating pillar strength such as Wilson (1973) and Barron (1984) had been replaced by refined numerical models.

Johnson et al. (2014) determined that the successful application of numerical models to estimate the strength of a coal pillar lies in providing the best available parameters and the pillar's response to loading. The authors stated that some empirical strength equations were unavailable for specific scenarios in boundary element method (BEM) programs such as LaModel (Heasley, 1998). Furthermore, Johnson et al. (2014) stated that the primary constraint for including the empirical strength equations into sophisticated BEM software is the lack of a function that defines the strength variation from the rib of the pillar within. Even though obtaining the gradient stress equations from empirical strength formulas is not trivial (Mark et al., 1992), the derivation of these equations initiates with two general assumptions:

- The stress within the yield zone is a function of the distance to the nearest rib boundary and is not dependant on the width of the pillar.
- The stress gradient within the yield zone does not change due to load nor time.
Mark and Iannacchione (1992) derived the gradient stress equation for the empirical strength equation proposed by Bieniawski (1968). First, the authors derived the gradient equation by calculating the ultimate load-bearing capacity of a square pillar using Bieniawski’s equation. Then obtained the pillar resistance increase by deriving the load-bearing capacity. Finally, after assuming that the vertical stress is a continuous function of the rib distance, Mark and Iannacchione (1992) derived the variable stress equation for Bieniawski’s empirical strength formula:

\[ \sigma_v = S_1 \left( 0.64 + 2.16 \frac{x}{h} \right) \]  

(2.14)

*Figure 2.11. Determination of pillar stress gradients from a pillar strength formula for a square coal pillar (Mark et al., 1992).*

In their study, Johnson et al. (2014) broadened Mark and Iannacchione's (1992) approach by proposing a methodology to obtain the variable stress equations given any empirical strength formula. Furthermore, the authors presented the application of these gradient functions by estimating different types of pillar cell elements used for BEM software. Initially, Johnson et al. (2014) used four general assumptions to obtain the gradient stress equations that make this methodology valid for deriving gradient stress equations from empirical strength formulas:

- The derivation of gradient stress equations is performed on square pillars.
- When the overall pillar strength reaches its maximum, all the "portions" of the pillar are at maximum strength.
- The variation of stress is a function of the distance to the nearest rib and is not dependant on the width of the pillar (Mark et al., 1992).
- The square pillar is divided into 8 symmetric pieces to simplify the calculations that relate the stress function to the failure force that is a function of its width.
Figure 2.1. Plan view of failure stress distribution of a square coal pillar subdivided into 8 symmetric pieces (Johnson et al., 2014).

The total vertical force $F(W)$ is calculated in terms of vertical pillar stress, then:

$$F(W) = 8 \int_0^W \int_0^y \sigma_v(x) \, dx \, dy = 8 \int_0^W I(y) \, dy$$  \hspace{1cm} (2.15)

Johnson et al. (2014) then used the Leibnitz Rule (1693) of fundamental integral calculus twice to equation (2.15):

$$\frac{dF}{dW} = 8I(W) = 8 \int_0^W \sigma_v(x) \, dx$$  \hspace{1cm} (2.16)

$$\frac{d^2F}{dW^2} = 8 \sigma_v(W)$$  \hspace{1cm} (2.17)

After obtaining equation (2.17), the general methodology developed by Johnson et al. (2014) was implemented as an initial test to Bieniawski’s strength formula (equation 2.8) to obtain the gradient stress equation.

For a square coal pillar with of dimension $w$, the load-bearing capacity is:

$$F = \sigma_p w^2 = \sigma_0 \left(0.64 + 0.36 \frac{w}{h}\right) w^2$$  \hspace{1cm} (2.18)

The authors defined the load-bearing capacity as $F = R$ and $w = 2W$ from figure 2.10. Then performing the differentiation, the vertical stress within the pillar, which is a linear function, is found:
\[
\frac{d^2 F}{8 dx^2} = \sigma_v(x) = \sigma_0 \left( 0.64 + 2.14 \frac{x}{h} \right)
\]  

(2.19)

In their research, Johnson et al. (2014) not only tested their proposed methodology for the obtention of variable stress equations on linear formulas like Bieniawski’s (1968), they tested their methodology for the Holland-Gaddy and Maleki pillar strength equation:

\[
\sigma_v(x) = 2.65\sigma_0 \left( \frac{x}{h} \right)^{0.5}
\]

Holland-Gaddy, (2.20)

\[
\sigma_v(x) = c_1 \{1 - e^{ax} [1 + 2ax + 0.5a^2x^2]\}
\]

Maleki, (2.21)

2.3.2. Corner Cell Equation

Modeling coal pillars in boundary element programs is allowed by the derivation of variable pillar strength equations (Johnson et al., 2014). Boundary Element Method software such as LaModel (Heasley, 1998) simulates a coal pillar strength by dividing the pillar into different types of coal cell elements. The two most common types of elements used in BEM software are rib and corner cell elements, where the rib element has a slightly higher strength than the corner element due to the extra confinement to which it is subjected (Johnson et al., 2014).

![Figure 2.13. Schematic of concentric material bands used for implementing Bieniawski’s coal properties in LaModel (Heasley, 1998).]

The rib cell, which is calculated by integrating the variable strength equation over the area of the rib cell (Johnson et al., 2014). As reported by Heasley (1998), the rib cell equation used in LaModel is:

\[
\sigma_v(x) = \sigma_0 \left( 0.64 + 2.14 \frac{x}{h} \right)
\]  

(2.22)
Where \( \bar{x} \) is the cell centroid

Johnson et al. (2014) declared that the second most common cell element used in BEM software is the corner cell element. Therefore, the variable strength equation is multiplied by the corner element area function, allowing the calculation of the force acting on one half of the corner element:

\[
F_{\text{corner}} = 2 \int_{x_1}^{x_2} \sigma_v(x) w(x)dx
\]

The width function of the corner element is \( w(x) \), where \( w \) is the element width (Johnson et al., 2014):

\[
w(x) = (x_2 - x)
\]

When

\[x = x_1, \text{ then } w(x) = x_2 - x_1 = w\]
\[x = x_2, \text{ then } w(x) = x_2 - x_1 = 0\]

In their study, Johnson et al. (2014) derived the corner cell equation initially for Bieniawski's variable strength equation by substituting it into the corner failure force equation:

\[
F_{\text{corner}} = 2 \int_{x_1}^{x_2} \sigma_0 \left( 0.64 + 2.14 \frac{x}{h} \right) (x_2 - x)dx
\]

After solving the integral and performing all the computations, the Bieniawski's corner cell equation is obtained (Johnson et al., 2014):

\[
\sigma_{\text{corner}} = \sigma_0 \left( 0.64 + \frac{2.16}{h} \left( \bar{x} - \frac{w}{6} \right) \right)
\]

Equation (2.25) is the same corner cell equation used by the coal wizard in the software LaModel (Heasley, 1998). Following the same approach for the obtention of the corner cell equation used for Bieniawski's strength equation, Johnshon et al. (2014) derived the corner cell elements for the Holland-Gaddy and Maleki's formulas.

Holland-Gaddy corner cell equation:

\[
\sigma_{\text{corner}} = \frac{\sqrt{2}}{h^{0.5}} \frac{\sigma_0}{w^2} \left[ \left( \bar{x} + \frac{w}{2} \right)^{2.5} - 2.5 \left( \bar{x} + \frac{w}{2} \right) \left( \bar{x} - \frac{w}{2} \right)^{1.5} + 1.5 \left( \bar{x} - \frac{w}{2} \right)^{2.5} \right]
\]

Maleki corner cell equation:
\[ \sigma_{\text{corner}} = c_1 \left\{ 1 + w^{-2} \left[ (2x_1x_2 - x_1^2 + a(x_2x_1^2 - x_1^3))e^{ax_1} - x_2^2e^{ax_2} \right] \right\} \] (2.28)

\[ C_1 = 3839 \text{ (structure) psi or 4700 (confinement) psi} \]

\[ C_2 = -0.26 \text{ (structure) psi or -0.339 (confinement) psi} \]

\[ a = \frac{2c_2}{h} \]
Chapter 3 Methodology

3.1. Introduction

As presented in the Literature Review section, Esterhuizen et al. (2011) developed an empirical equation to estimate a stone pillar strength (equation 3.1). This equation is a power equation that is dependent on the shape and volume of the stone pillar:

\[ S = k \left( \frac{w^{0.3}}{h^{0.59}} \right) \]  \hspace{1cm} (3.1)

In this research, the gradient strength equation for the stone mine pillar equation (3.1) was derived following similar approaches to those presented by Mark et al. (1992) and Johnson et al. (2014). In these approaches, it is assumed that the variation of stresses within the pillar is a function of distance to the closest rib. Following a similar procedure to the one used by Johnson et al. (2014), the square pillar is first divided into eight symmetric pieces for simplicity in calculations (Figure 3.1). Then, cell strength equations were derived for Esterhuizen's base empirical stone mine pillar strength equation (Esterhuizen et al., 2011).

![Figure 3.1. Plan view of the stress distribution of a square stone pillar (After Johnson et al., 2014).](image-url)
3.1.1. Stone Pillars Variable Strength Equation Derivation

Equation 3.1. is the empirical strength equation for stone pillars developed by Esterhuizen et al. (2011) where; \( \sigma_p \) is the average pillar strength, \( \sigma_0 \) is the rock strength parameter (it can be expressed in terms of the Unconfined Compressive Strength (UCS)), \( W \) is the width and \( h \) is the height of the pillar.

\[
\sigma_p = \sigma_0 \times \frac{W^{0.3}}{h^{0.59}} \quad (3.2)
\]

The ultimate load-bearing capacity of the pillar (Force) is calculated by multiplying the pillar strength \( \sigma_p \) by the area of the pillar (Johnson et al., 2014):

\[
\sigma_p \times W^2 = Force = F \quad (3.3)
\]

Then performing the substitution of equation (3.2) into equation (3.3):

\[
F = \sigma_0 \times \frac{W^{0.3}}{h^{0.59}} \times W^2 \quad (3.4)
\]

\[
F = \sigma_0 \times \frac{W^{2.3}}{h^{0.59}} \quad (3.5)
\]

Following the statement by Mark and Iannachionne (1992), the maximum value of horizontal location within the pillar is then \( x = \frac{W}{2} \) (Figure 3.1), \( 0 < x < w \). Then substituting into equation (3.4):

\[
F = \sigma_0 \times \left( \frac{2x}{h^{0.59}} \right)^{2.3} \quad (3.6)
\]

\[
F = 4.92 \times \sigma_0 \times \frac{x^{2.3}}{h^{0.59}} \quad (3.7)
\]

As presented in the literature review section, Johnson et al. (2014) defined the total vertical force \( (F) \) in terms of the vertical pillar stress \( (\sigma_v) \) for a square pillar as:

\[
F(W) = 8 \int_0^W \int_0^y \sigma_v(x) \, dx \, dy = 8 \int_0^W l(y) \, dy \quad (3.8)
\]

Where \( l(y) = \int_0^y \sigma_v(x) \, dx \)  \( (3.9) \)
Moreover, Johnson et al. (2014) used Leibnitz Rule (1693) twice in equation (3.8) to obtain the total vertical force in terms of the vertical pillar stress for a square pillar:

\[ \frac{d^2 F}{dW^2} = 8 \sigma_v(W) \quad (3.10) \]

Consequently, using the methodology determined by Johnson et al. (2014) into the square stone pillar variables, equation (3.11) and equation (3.12) are obtained:

\[ \frac{dF}{dw} = 8 \int_0^w \sigma_v(x) \, dx \quad (3.11) \]
\[ \frac{d^2 F}{dW^2} = \frac{d^2 F}{dx^2} = 8 \sigma_v(x) \quad (3.12) \]

Then the vertical pillar stress of a square stone pillar is obtained by substituting equation (3.7) into equations (3.11) and (3.12):

\[ \frac{dF}{dx} = 4.92 \,(2.3) \sigma_0 \frac{x^{1.3}}{h^{0.59}} = 11.33 \sigma_0 \frac{x^{1.3}}{h^{0.59}} \quad (3.13) \]
\[ \frac{d^2 F}{dx^2} = 11.33 \,(1.3) \sigma_0 \frac{x^{0.3}}{h^{0.59}} = 14.73 \sigma_0 \frac{x^{0.3}}{h^{0.59}} \quad (3.14) \]

Finally, the second derivative of the load-bearing capacity is divided by the eight symmetric pieces of the pillar to obtain the cell strength equation for stone mine pillars:

\[ \sigma_v(x) = 1.84 \, \sigma_0 \frac{x^{0.3}}{h^{0.59}} \quad (3.15) \]

Where, \( \sigma_v(x) \) is strength at distance \( x \) from the rib of the stone pillar, "\( \sigma_0 \)" is the rock strength parameter expressed in terms of the UCS, "\( x \)" is the distance from the rib towards the center of the pillar, and "\( h \)" is the height of the pillar.

Equation (3.15) can be expressed both in the International System of Units (S.I.) and Imperial System of Units (I.U.). For pillar dimensions in the I.U. system, the rock strength parameter "\( \sigma_0 \)" is computed as 0.92 xUCS, for the S.I. "\( \sigma_0 \)" becomes 0.65 xUCS (Esterhuizen et al., 2011).
3.1.2. Stone Pillars Corner Cell Equation Derivation

Johnson et al. (2014) stated that the second most common cell element used to model pillars in boundary element program is the corner cell element (Figure 3.2).

Figure 3.2. Stress Profile for different types of elements in a square stone pillar (After Johnson et al., 2014).

Mark et al. (1992) defined the ultimate load-bearing capacity equation for corner cell elements as:

$$ F_{\text{corner}} = 2 \int_{x_1}^{x_2} \sigma_v(x) w(x) \, dx $$  \hspace{1cm} (3.16)

Where \( w(x) \) is the width function for square stone pillars and "W" is the cell width:

$$ w(x) = (x_2 - x) $$  \hspace{1cm} (3.17)

When

- \( x = x_1 \), then \( w(x) = x_2 - x_1 = W \)
- \( x = x_2 \), then \( w(x) = x_2 - x_1 = 0 \)

Substitution of the cell strength equation (equation 3.15) and equation (3.17) into the Corner Cell Force Equation (3.16) gives:

$$ F_{\text{corner}} = 2 \int_{x_1}^{x_2} \sigma_0 \frac{x^{0.3}}{h^{0.59}} (x_2 - x) \, dx $$  \hspace{1cm} (3.18)

For simplicity, the constants of equation (3.18) are taken out of the integration and grouped as one:
Consequently, equation (3.18) becomes:

\[ F_{\text{corner}} = 2 \, c_1 \int_{x_1}^{x_2} x^{0.3} (x_2 - x) \, dx \]  

(3.20)

Separating \( c_1 \) from the integration:

\[ \frac{F_{\text{corner}}}{2 \, c_1} = \int_{x_1}^{x_2} x^{0.3} (x_2 - x) \, dx \]  

(3.21)

From the linearity rule of integral calculus, equation (3.21) becomes:

\[ \frac{F_{\text{corner}}}{2 \, c_1} = \int_{x_1}^{x_2} x^{0.3} \, dx - \int_{x_1}^{x_2} x^{1.3} \, dx \]  

(3.22)

Consequently, equation (3.22) turns into:

\[ \frac{F_{\text{corner}}}{2 \, c_1} = x_2 \int_{x_1}^{x_2} x^{0.3} \, dx - \int_{x_1}^{x_2} x^{1.3} \, dx \]  

(3.23)

Then, solving the integration in equation (3.23):

\[ \frac{F_{\text{corner}}}{2 \, c_1} = \frac{x_2}{1.3} \left( x_1^{1.3} \right) - \frac{1}{2.3} \left( x_2^{2.3} \right) \right|_{x_1}^{x_2} \]  

(3.24)

\[ \frac{F_{\text{corner}}}{2 \, c_1} = \frac{x_2}{1.3} \left( x_1^{1.3} - x_1^{1.3} \right) - \frac{1}{2.3} \left( x_2^{2.3} - x_1^{2.3} \right) \]  

(3.25)

\[ \frac{F_{\text{corner}}}{2 \, c_1} = \frac{x_2}{1.3} \left( x_1^{1.3} \right) - \frac{x_2}{1.3} \left( x_1^{1.3} \right) - \frac{2.3}{2.3} \left( x_2^{2.3} \right) + \frac{2.3}{2.3} \left( x_1^{2.3} \right) \]  

(3.26)

\[ \frac{F_{\text{corner}}}{2 \, c_1} = \frac{x_2^{2.3}}{1.3} - \frac{x_2^{1.3}}{1.3} - \frac{x_2^{2.3}}{2.3} + \frac{x_1^{2.3}}{2.3} \]  

(3.27)

Solving for the like terms:

\[ \frac{F_{\text{corner}}}{2 \, c_1} = \frac{x_2^{2.3}}{2.99} - \frac{x_2^{1.3}}{2.99} + \frac{x_1^{2.3}}{2.3} \]  

(3.28)

Taking \( \frac{1}{2.99} \) as a common divisor from the right side of the expression in equation (3.28):

\[ \frac{F_{\text{corner}}}{2 \, c_1} = \frac{1}{2.99} \left( x_2^{2.3} - 2.3 \, x_2^{1.3} + 1.3 \, x_1^{2.3} \right) \]  

(3.29)
Then clearing the $F_{\text{corner}}$ variable from equation (3.29):

$$F_{\text{corner}} = \frac{2 c_1}{2.99} (x_2^{2.3} - 2.3 x_2(x_1^{1.3}) + 1.3 x_1^{2.3}) \quad (3.30)$$

Subsequently, following Newton's Laws of Motion where $\text{Force} = \text{area} \times \text{stress}$:

$$\sigma_{\text{corner}} = F_{\text{corner}} W^2 \quad (3.31)$$

Replacing the stress relation (equation 3.31) into equation (3.30):

$$\sigma_{\text{corner}} = \frac{F_{\text{corner}}}{W^2} = \left( \frac{2 c_1}{2.99 W^2} \right) (x_2^{2.3} - 2.3 x_1^{1.3} x_2 + 1.3 x_1^{2.3}) \quad (3.32)$$

Jhonson et al. (2014) stated that $x_1$ and $x_2$ need to be in terms of the width for the $\sigma_{\text{corner}}$ to be a function of the width. The authors proposed a relation for the square pillar, where the locations of $x_1$ and $x_2$ are written in terms of the average location of the pillar $\bar{x}$ (figure 3.3).

![Figure 3.3. Average location within the square stone pillar (After Jhonson et al., 2014).](image)

Performing the substitution of $x_1$ and $x_2$ relations into equation (3.32):

$$\sigma_{\text{corner}} = \left( \frac{2 c_1}{2.99 W^2} \right) \left[ (\bar{x} + \frac{W}{2})^{2.3} - 2.3 (\bar{x} + \frac{W}{2}) (\bar{x} - \frac{W}{2})^{1.3} + 1.3 (\bar{x} - \frac{W}{2})^{2.3} \right] \quad (3.33)$$
Replacing $c_1$ from equation (3.19) into equation (3.33):

$$
\sigma_{\text{corner}} = \left( \frac{2(1.84 \sigma_0)}{2.99 w^2 h^{0.59}} \right) \left[ \left( \bar{x} + \frac{W}{2} \right)^{2.3} - 2.3 \left( \bar{x} + \frac{W}{2} \right) \left( \bar{x} - \frac{W}{2} \right)^{1.3} + 1.3 \left( \bar{x} - \frac{W}{2} \right)^{2.3} \right] (3.34)
$$

Finally, solving the numerical values in equation (3.33), the Stone Pillar Corner Cell Equation is obtained:

$$
\sigma_{\text{corner}} = \left( \frac{1.23 \sigma_0}{h^{0.59} W^2} \right) \left[ \left( \bar{x} + \frac{W}{2} \right)^{2.3} - 2.3 \left( \bar{x} + \frac{W}{2} \right) \left( \bar{x} - \frac{W}{2} \right)^{1.3} + 1.3 \left( \bar{x} - \frac{W}{2} \right)^{2.3} \right] (3.35)
$$

Where $\sigma_{\text{corner}}$ is the average stress at the corner element of the pillar, "$\sigma_0$" is rock strength parameter-dependent of the UCS, "$\bar{x}$" is the average location within the pillar, "$W$" is the width of the pillar and "$h$" is the height of the pillar. Similarly to the empirical strength equation for stone mine pillars (Esterhuizen et al., 2011), the Stone Pillar Corner Cell Equation can be calculated in the International System of Units by expressing "$\sigma_0$" as 0.65 x UCS, and in the Imperial System of Units by expressing "$\sigma_0$" as 0.92 x UCS.

### 3.2. Verification Equations against S-Pillar Equation

Once both cell strength equations for stone mine pillars have been derived, it is necessary to test them against the empirical S-Pillar equation. Therefore, the pillar geometry and rock mass parameters for testing the equations were selected from the S-Pillar database (Esterhuizen et al., 2011).

Initially, to verify the variable strength equations, the average UCS value for stone mines was selected. Afterward, the pillar dimensions were selected as the average values for pillar widths and heights from the case history mines in S-Pillar database (Esterhuizen et al., 2011). The number and types of the different in-seam materials (elements) were set using the parameters of the geometry of the pillar and a standard element size of 5 ft in the LaModel preprocessor LamPre (Heasley, 1998: 2009).

As stated by Esterhuizen et al. (2011), the median value of UCS in the Eastern and Midwestern United States stone mines falls in the range of 11,900 to 30,000 psi. Therefore, the average UCS value within this range (20,950 psi) was selected for this initial verification. First, the verification for the variable strength equations is performed for a square pillar with a width-to-height ratio of
1. As Roberts et al. (2007) stated, the pillar's strength was dependent on the width of the pillar and was affected by its height. Therefore the width of the pillar is the first geometry parameter selected for testing the cell strength equations derived in this thesis. Following the stone mine pillar dimensions obtained by Esterhuizen et al. (2011), it was identified that the average width of pillars in the S-Pillar database ranges between 40 and 43 feet. Consequently, the pillar dimensions for the initial verification are assumed to be squared with 40 ft wide and 40 ft height. Table 3.1 shows the selected pillar parameters.

**Table 3.1. Stone Pillar Parameters for a width-to-height ratio of 1.**

<table>
<thead>
<tr>
<th>Stone Pillar Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pillar Width [ft]</td>
</tr>
<tr>
<td>Pillar Height [ft]</td>
</tr>
<tr>
<td>Average Stone UCS [psi]</td>
</tr>
</tbody>
</table>

As the geometric parameters and the limestone rockmass UCS are selected for the initial verification, the LaModel preprocessor LamPre is used to obtain the type and number of the pillar's different in-seam materials (elements). A total of 9 in-seam materials were set to obtain the proper element distribution for 40 ft pillar and 5 ft element size.

![Figure 3.4. Selection of the number of in-seam materials in LamPre.](image)

Once the number of in-seam materials and the element sizes are obtained, the grid function of LamPre is used to generate the correct distribution of the different elements for the pillar. As a
result, the pillar was built with three different rib elements D-F-H and four different corner elements C-I-G-E (Figure 3.5). The element A in LaModel (Heasley 1998; 2009) is not used in the stone pillar construction since element A is Linear Elastic by default, and the stone pillar behavior is Elastic Plastic. Then, using the cell strength equations, the peak stress for each element of the pillar was computed.

To calculate the total strength of the pillar, the stress obtained for each element was multiplied by the area of the element. This gives the load-bearing capacity for each element. Then the load-bearing capacities for all the elements are summed up to obtain the total load-bearing capacity of the pillar. Finally, the total load-bearing capacity of the pillar is divided by the entire area of the pillar to compute the average strength of the pillar.
To verify the cell strength functions, strengths computed by the empirical $S_P$-Pillar equation and cell functions were compared in Table 3.2.

Table 3.2. Strength comparison for a pillar with a width-to-height ratio of 1.

| Area of the Element [ft$^2$] | 25 |
| Area of the Pillar | 1,600 |
| Stone Strength [psi] | 20,950 |
| Width of the Pillar [ft] | 40 |
| Height of the Pillar [ft] | 40 |
| Cell Pillar Strength Functions [psi] | 6618 |
| Esterhuizen Pillar Strength [psi] | 6613 |
| Strength Difference | 0.082 % |

For a pillar with a width-to-height ratio of 1, the difference in strength is 0.082% or 5 psi, as shown in Table 3.2. Once the variable strength equations have been tested for the average values of stone pillars, they are verified for pillars with different element sizes and width-to-height ratios.

### 3.2.1. Different width-to-height ratio

The pillar's dimensions were changed to keep the same element size (5 ft) to obtain different width-to-height ratios. The $S$-pillar database developed by Esterhuizen et al. (2011) shows that the width-
to-height of 83% of the pillars surveyed fit in the range of 0.5 to 2.0. Therefore, the verification was done for pillars with a width-to-height ratio of 0.5, 1, 1.5, and 2.0.

The pillar's width is set to 40 ft, and the height is varied. Then, the total strength of the pillars using the cell strength equations and Esterhuizen's equation were compared for the different width-to-height ratios following the same methodology employed to perform the initial verification. In table 3.3. and Figure 3.7, the results of the verification for different width-to-height ratios are summarized.

### Table 3.3. Comparison of the pillar strength for different width-to-height ratios.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.500</td>
<td>40</td>
<td>80</td>
<td>4397</td>
<td>4393</td>
<td>0.0822%</td>
</tr>
<tr>
<td>1.000</td>
<td>40</td>
<td>40</td>
<td>6618</td>
<td>6613</td>
<td>0.0822%</td>
</tr>
<tr>
<td>1.500</td>
<td>40</td>
<td>27</td>
<td>8407</td>
<td>8400</td>
<td>0.0822%</td>
</tr>
<tr>
<td>2.000</td>
<td>40</td>
<td>20</td>
<td>9962</td>
<td>9954</td>
<td>0.0822%</td>
</tr>
</tbody>
</table>

*Figure 3.7. Pillar strength width-to-height comparison.*
It is observed that for the different width-to-height ratios, the difference between the cell strength equations and the empirical strength equation developed by Esterhuizen et al. (2011) is constant at 0.082%.

3.2.2. Different element size

Once the variable strength equations have been tested for different width-to-height ratios, the behavior of the equation for different element sizes is also verified by keeping the width-to-height ratio and the pillars’ width constant at 1 and 50ft respectively, and changing the element size (2, 2.5, 5, and 10 feet).

Esterhuizen et al. (2011) stated that the width of underground stone pillars in the Eastern United States ranges between 15 ft and 70.5 ft. Therefore, the width of the pillars selected for different size elements fits in the range provided. Moreover, this width was selected since it made selecting the yielding elements in the LaModel preprocessor LamPre (Heasley 1998;2009) easier.

For testing the cell strength equations for a pillar composed of 2.0 ft elements, 24 in-seam materials were defined. For the pillars with elements of 2.5 ft, 20 different yield elements were defined in LamPre. 10 different in-seam elements were defined for an element size of 5ft. Finally, for testing the cell strength equations for elements of 10 ft, the number of yield materials defined was 5.

Table 3.4. Comparison of pillar strength for different element sizes for pillars with w:h of 1.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>50.00</td>
<td>50.00</td>
<td>2.00</td>
<td>6196</td>
<td>6198</td>
<td>0.026%</td>
</tr>
<tr>
<td>50.00</td>
<td>50.00</td>
<td>2.50</td>
<td>6200*</td>
<td>6198</td>
<td>0.028%</td>
</tr>
<tr>
<td>50.00</td>
<td>50.00</td>
<td>5.00</td>
<td>6200*</td>
<td>6198</td>
<td>0.035%</td>
</tr>
<tr>
<td>50.00</td>
<td>50.00</td>
<td>10.00</td>
<td>6162</td>
<td>6198</td>
<td>0.573%</td>
</tr>
</tbody>
</table>

*The value is the same considering the round up without decimals, however for 2.5 ft elements the strength is 6200.012 psi, and for 5 ft elements is 6200.444 psi.
As observed in Table 3.4, for smaller elements, there is a smaller difference between the cell strength equations and the S-Pillar equation (Esterhuizen et al., 2011). This behavior for smaller and larger elements can be observed in the LaModel calibration file developed by Heasley (2010).

3.3. Post-peak stress behavior of the cell strength equations

Once the cell strength equations are derived and tested against the S-Pillar equation (Esterhuizen et al., 2011), the stress-strain behavior curves of the derived equations are presented. As shown in the Literature Review section, the post-peak stress behavior of stone mine pillars has been presented as progressive failure mechanisms (Esterhuizen et al., 2011; Iannachione, 1999; Suner, 2021). Moreover, numerical modeling does not provide a totally realistic idea of stone pillar behavior (Iannachione, 1999). Therefore, while numerical modeling produces a useful means to predict the post-peak stress behavior, the actual behavior is not entirely known.

In this research, the post-peak stress behavior for the cell strength equations is assumed constant once the peak stress is reached. For the pillars with width-to-height ration less than 2, strain softening post peak behavior would be expected. However, there isn’t any rational method or stress measurement results available for stone mine pillar to estimate the true in situ strain softening behavior. Elastic perfectly plastic post peak behavior assumption used in this study might not simulate the load transferred from failed rib elements to the pillar core accurately, however overall strength of the pillar can be simulated accurately. For the derivation of these curves, the cell stress values are calculated using the same values for the initial verification of the cell strength equations, as can be observed in table 3.5. The element strain is computed using the peak stress of the element calculated using the cell strength equations and the limestone elastic modulus.

| Table 3.5. Pillar parameters of stress-strain curves for the cell strength equations. |
|---------------------------------|---------|
| Area of the Element [ft$^2$]   | 25      |
| Area of the Pillar              | 1,600   |
| Stone Strength [psi]            | 20,950  |
| Width of the Pillar [ft]        | 40      |
| Height of the Pillar [ft]       | 40      |
| Average Stone Elastic Modulus [psi] | 6,236,622 |
| Number of Rib Elements          | 4       |
| Number of Corner Elements       | 4       |
Table 3.6. Stress-strain values for the different types of elements.

<table>
<thead>
<tr>
<th>Element</th>
<th>Stress [psi]</th>
<th>Strain</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>5296</td>
<td>0.000849</td>
</tr>
<tr>
<td>F</td>
<td>7364</td>
<td>0.001181</td>
</tr>
<tr>
<td>D</td>
<td>8583</td>
<td>0.001376</td>
</tr>
<tr>
<td>B</td>
<td>9495</td>
<td>0.001522</td>
</tr>
<tr>
<td>I</td>
<td>4361</td>
<td>0.000699</td>
</tr>
<tr>
<td>G</td>
<td>6123</td>
<td>0.000982</td>
</tr>
<tr>
<td>E</td>
<td>7807</td>
<td>0.001252</td>
</tr>
<tr>
<td>C</td>
<td>8905</td>
<td>0.001428</td>
</tr>
</tbody>
</table>

Figure 3.8. Stone stress-strain curves.
Chapter 4 Case Study Implementation

4.1. Introduction

In this chapter, the stone mine pillar cell strength equations are integrated in the LaModel input file to obtain the stress and safety factor analysis of a case study mine in the Loyalhanna Limestone formation. First, the general geology of the Loyalhanna Formation in southwestern Pennsylvania and northcentral West Virginia is presented. Then, the overburden stress, total vertical stress, and pillar stress safety factor plots for the case study mine using the LaModel program and the cell strength equations are presented. Finally, the results from LaModel are compared to the ones obtained using the S-Pillar software (Esterhuizen et al., 2011) for the case study mine.

4.2. Case Study Mine

The stone pillar cell strength equations implemented in to LaModel are tested in an successful underground stone case study mine. For modeling the underground stone mine in LaModel using the cell strength equations, the dimensions and parameters of the pillars, and UCS of the Loyalhanna Limestone formation were obtained from the mine personnel (Table 4.1).

<table>
<thead>
<tr>
<th>General Mine Information</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Width of pillars [ft]</td>
<td>50</td>
</tr>
<tr>
<td>Height of Pillars [ft]</td>
<td>27</td>
</tr>
<tr>
<td>Average Stone Strength [psi]</td>
<td>29401.01</td>
</tr>
</tbody>
</table>

Mine layout is shown in Figure 4.2. Also, from the mine information, there are two main benched areas. It is important to note that in this case study mine, pillar layout is design very successfully and both the development and benched pillars are stable and in good conditions. Thus, it 17 yield zones are selected to accommodate the effect of different height pillars on the strength calculation. The development pillars have a width-to-height ratio of 1.85, where the width is 50 ft, and the height is 27 ft. The height increases to 50 ft for benched pillars, and the width-to-height ratio becomes 1.0.
4.2.1. Loyalhanna Limestone formation

The Loyalhanna Limestone formation is a nearly horizontal to gently folded sedimentary unit that belongs to the Appalachian Basin, located in Northcentral West Virginia, Southern Pennsylvania, Maryland, and Ohio (Gallagher, 1984; Iannachione et al., 2002). According to Kerrigan (2016), the Appalachian plateau is the westernmost province of the Appalachian mountain belt. Furthermore, Edmunds et al. (1979) stated that The Loyalhanna Limestone is a lower member of the Mississippian-age Mauch Chunk Formation and is characterized by broad folding with dips ranging from 20 to less than 5 degrees. Moreover, the Loyalhanna Limestone has an average thickness of 60 ft and has a maximum thickness of 103 ft and is described as a very "thin sheet" body (Adams et al., 1970).

Adams et al. (1970) stated that the Loyalhanna formation is composed of a variation of quartzose limestone and calcareous quartz arenite. The Loyalhanna Limestone is a unit typically located deeper than a thousand feet from the surface, however it crops out at numerous southwestern Pennsylvania and northeastern West Virginia sites (Iannacchione et al., 2002). Iannacchione et al. (2002) stated that the outcrops of the Loyalhanna Limestone had been historically utilized for
aggregate mining. In addition, Iannacchione et al. (2002) determined that most of the exposures of this unit occur along the crest of Chestnut Ridge, Laurel Hill, and Negro Mountain Anticlines. Esterhuizen et al. (2018) determined that, while typically the Loyalhanna Limestone is mined at depths in the region, it is also mined in several locations where folding has brought the formation closer to the surface.

The Loyalhanna Limestone is a significant petroleum and stone-producing unit in the Appalachian Basin (Ahlbrandt, 1995; Iannacchione et al., 2002). Iannacchione et al. (2002) stated that this unit is rated as a super pavement aggregate by the Pennsylvania Department of Transportation, enhancing its importance and the necessity of mining operations to extract it. In 2002 Iannacchione et al. stated that the Loyalhanna Limestone unit reveals a complex structural environment. The authors elaborated that knowing the characteristics, such as dip, orientation, and spacing of the Loyalhanna Anticlinals, is necessary for mine design and development. The complex geologic features in the Loyalhanna Limestone unit are indicated as potential reason for roof falls and rib spalling in the underground limestone mines that operate on it, leading to injuries (Iannacchione et al., 2002; Esterhuizen et al., 2018).

4.2.2. Underground limestone case study mine in LaModel

For testing the cell strength equations for the stone mine stress gradient derived and implemented into LaModel in this research, operational data such as a topographic map, roof elevation data, mine map, and Uniaxial Compressive Strength (UCS) data was collected from the case study mine. The case study mine is located at the Loyalhanna Limestone formation, where it dips at less than 5°. Thus, it is considered a flat-lying mine.

As presented in Table 4.1., the mine pillar dimensions are 50 ft x 50 ft. The development mining height is 27 ft, and the total pillar height after floor benching is 50 ft. The UCS laboratory results provided by the mine are averaged to 29,401 psi. This is a value expected for the Loyalhanna Limestone that is usually 29,000 psi to 32,000 psi. The width-to-height ratio of the pillars in the development stage (1.85) and fully benched (1.00) are larger than the minimum suggested value of 0.8 (Esterhuizen et al., 2011).

The topography elevation grid and the seam grid surface were generated from the elevation contours and the roof elevation contours provided by the mine. For generating the grid files, the
Stability-Mapping program was used. The Stability-Mapping program was developed to combine geologic and stress information to create a map of the foreseen stability of the mine (Wang, 2005; Wang and Heasley, 2006; Heasley et al., 2018). In addition, Stability-Mapping incorporates the output from the LaModel program for stress input. The overburden grid was generated by subtracting the seam top elevation grid values from the surface elevation grid values. With the elevation grid and the seam grid obtained from the Stability-Mapping analysis, the cell strength equations are implemented for the case study mine in LaModel.

![Mine topographic layout](image)

*Figure 4.2. Mine topographic layout.*

To implement the cell strength equations in the case study mine, two initial conditions were determined. First, since the pillars were 50 ft x 50 ft, the element size selected was 5 ft. Then, the number of in-seam materials for the whole mine is determined. From the element and pillar size, a total of 8 in-seam elements are defined for each pillar, 4 rib cell elements, and 4 corner cell elements. Given that the input parameters for calculating the benched pillars are different than the development pillars, a different set of 8 in-seam materials for benched pillars is also defined in LaModel. Thus, 17 in-seam materials are defined for the whole mine grid, including development
and benched pillars. For the development pillars, the rib elements are P-N-L-J, and the four corner elements are Q-O-M-K. Furthermore, the rib elements are H-F-D-B for the benched pillars, and the corner elements are I-G-E-C, according to the element distribution in the LaModel preprocessor LamPre. In Figure 4.3, the different types of elements used to simulate development and benched pillars are demonstrated.

Figure 4.3. Element distribution for development and benched pillars in LaModel.

As the different types and number of in-seam materials are defined, the cell strength equations are used to obtain the peak stress of each of the different elements of the development and benched pillars. The peak stress results from the different elements are manually added into the In-Seam Material Models option (Figure 4.4.) of the LaModel preprocessor LamPre (Heasley, 1998). Furthermore, the In-Seam material type selected for the different rib and corner elements for the case study mine is Elastic-Plastic for intact material (Figure 4.4).
As the peak stress for all the elements is introduced into LamPre, the grid option of the preprocessor is used to apply the yield zone elements for the whole mine grid. This option changes the different elements in the grid following the previously defined yield zone. Initially, the pillars in the LamPre grid have the element distribution for the development pillars only due to the large number of yield materials defined compared to the number of elements per pillar in the mine grid. Therefore, the elements for benched pillars are manually introduced in the grid following the correct element distribution from the development pillars (Figure 4.5).
Once the mine grid has the correct yield element distribution and peak stress values obtained with the cell strength equations in LamPre for development and fully benched pillars, the file is saved as the input file for LaModel. After the input file for the case study mine is created, the LaModel program is run to calculate the seam's stress and displacements (Heasley, 1998). It is important to note that the input file created in the preprocessor LamPre needs to be stored in the same folder where the topography file is stored for a proper seam stress calculation. Finally, the output from the calculation phase is stored as a data file for subsequent analysis by the post-processing program LamPlt. Using the post-processing program LamPlt, the different stress items available in LaModel are presented in the case study mine grid. The stress items obtained for the case study mine in the Loyalhanna formation using the Boundary Element Method software LaModel are the overburden stress distribution, the total vertical stress on the seam, and stress safety factor of the stone mine pillars.

![Overburden Stress](image)

*Figure 4.6. Overburden stress distribution (psi).*
Figure 4.7. Vertical stresses on the seam (psi).
It is observed that the benched pillars and "edge" pillars surrounding the fully benched pillars present a lower Factor of Safety (FOS) than the development pillars. Moreover, the total vertical stress on the seam is incremented in pillars with higher overburden stress. Therefore, in the deepest areas of the mine, the factor of safety is smaller, and the vertical stress is higher, disregarding if the pillars are development or are fully benched.

Final analysis that is conducted in this thesis is the simulation of the benched pillars with a reduced stiffness. The benched pillars strain was reduced to ½ of the original strain. Then the model was run again to obtain the stress variation in the benched and perimeter pillars. The total vertical stress of the mine with the reduced stiffness is observed in Figure 4.9.
Figure 4.9. Total vertical stress with reduced stiffness in benched pillars.

From Figure 4.9, it is observed that the benched pillars in the lower areas of the mine have higher stresses with reduced stiffness. Additionally, it is observed that there is an increment in the perimeter pillars of benched areas. Similar observations are also made by Esterhuizen et al. (2011).

4.2.3. S-Pillar Comparison

Once the LaModel plots for the overburden stress distribution, vertical stress on the seam, and stress safety factor using the cell strength equations are obtained, the software S-Pillar is used to compare pillar stress and safety factor results for the case study mine.

From Figure 4.6, the maximum overburden stress contour is 760 psi. This value is used to calculate the maximum overburden depth by assuming a 1.125 psi/ft vertical stress gradient.
Maximum Overburden Depth = \frac{760 \text{ psi}}{1.125 \text{ psi/ft}} = 675 \text{ ft} \quad (4.1)

The Maximum Overburden Depth calculated from the overburden stress distribution from LaModel is used as an input parameter in the S-Pillar (Esterhuizen et al., 2011). The S-Pillar uses the maximum overburden depth of the mine to obtain the safety factor results on the benched and development pillars. The input parameters and results of S-Pillar analysis for the case study mine are presented in Table 4.2. In Figure 4.9, the FOS result of the mine is presented.

Table 4.2. Input parameters and results of S-Pillar analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pillar Width</td>
<td>50 ft</td>
</tr>
<tr>
<td>Pillar Length</td>
<td>50 ft</td>
</tr>
<tr>
<td>Maximin Overburden Depth</td>
<td>675 ft</td>
</tr>
<tr>
<td>Pillar Height (Development)</td>
<td>27 ft</td>
</tr>
<tr>
<td>Pillar Strength (Development)</td>
<td>12,441 psi</td>
</tr>
<tr>
<td>Pillar Height (after Benching)</td>
<td>50 ft</td>
</tr>
<tr>
<td>W:H (Development)</td>
<td>1.85</td>
</tr>
<tr>
<td>FOS (Development)</td>
<td>4.17</td>
</tr>
<tr>
<td>W:H (after Benching)</td>
<td>1</td>
</tr>
<tr>
<td>Pillar Strength (after benching)</td>
<td>8,649 psi</td>
</tr>
<tr>
<td>LDF</td>
<td>1</td>
</tr>
<tr>
<td>UCS</td>
<td>29,401 psi</td>
</tr>
<tr>
<td>FOS (after benching)</td>
<td>2.9</td>
</tr>
</tbody>
</table>
From the results of S-Pillar analyses, the NIOSH design guidelines recommendations are satisfied as follows:

1. The factor of safety of pillars during development, 4.17, and after benching, 2.90, are larger than the minimum recommended FOS of 1.9.
2. Width-to-height ratio of pillars during development, 1.85, and after benching, 1 are larger than the minimum suggested value of 0.8.

Furthermore, the FOS for pillars using S-Pillar analysis is smaller than the results obtained using LaModel. This is because S-Pillar uses the maximum overburden depth to do the calculation while using LaModel, the stress calculations are done using the different overburden values for the whole mine, as shown in the problem statement section.
5.1 Summary

In the U.S., an empirical pillar strength equation and the S-Pillar program were developed to assist in the design of stable pillars for room-and-pillar workings in underground stone mines (Esterhuizen et al., 2011). By default, the S-Pillar program calculates the pillar load as the maximum depth over the pillar layout, and the stress calculation is only valid if the areas of the mine use regular-sized pillars (Esterhuizen et al., 2011). This research aims to extend the application of the S-Pillar strength equation to the Boundary Element Method software (BEM) LaModel (Heasley, 1998) to allow stone mine operators to integrate accurate overburden stress distribution under variable topography and to better assess the global stability of their pillar systems. This objective is achieved by the derivation of a stress gradient equation for stone pillars from the empirical pillar strength equation proposed by Esterhuizen et al. (2011).

The gradient stress equation for stone mines is derived by following a similar approach to those presented by Mark et al. (1992) and Johnson et al. (2014), where it is assumed that the variation of stresses within the pillar is a function of the shortest distance to the rib boundary. Following Johnson et al.’s (2014) study, the gradient stress equation for stone pillars is obtained for rib and corner cell elements. As the cell pillar strength functions are derived, they were tested against the S-Pillar equation (Esterhuizen et al., 2011). Pillar dimensions and rock mass parameters for verifying the equations relative to S-Pillar strength estimates were selected from the S-Pillar database (Esterhuizen et al., 2011). Then, the behavior of the equations is verified for different width-to-height ratios and different element sizes. The verification of the derived gradient stress equations against the empirical S-Pillar equation (Esterhuizen et al., 2011) is done for pillars with different width-to-height ratios (0.5, 1.0, 1.5, and 2.0) and different element sizes (2.0, 2.5, 5.0, and 10.0 ft).

After the verification, the cell pillar strength equations are implemented for a case study mine in the Loyalhanna Limestone formation. The equations are used to calculate the stress of the different elements defined in LamPre for the case study mine. These stresses are employed as an input parameter in the In-Seam Materials Mode option in LamPre for the different elements. Finally, the overburden stress distribution, pillar safety factors, pillar vertical stress distribution, and the other...
output stress items formats available in LaModel (Heasley, 1998) are obtained using the cell strength functions for the case study mine.

5.2. Conclusions

The derived cell strength equations for stone mine pillars are tested against the S-Pillar strength equation for different scenarios. Initially, using a fixed element size of 5 feet, the cell strength equations are verified for pillars with width-to-height ratios of 0.5, 1.0, 1.5, and 2.0, where it is observed that for these cases, the difference between the cell strength equations and the empirical strength equation, developed by Esterhuizen et al. (2011) is constant at $0.082\%$. The difference between the derived equation and the empirical strength equation (Esterhuizen et al., 2011) shows that the behavior of the derived equations is very similar to the one presented by the empirical S-Pillar strength equation. The reason of the difference between empirical and LaModel estimated pillar strengths was due to the numerical approximation of the pillar strength in the LaModel. Moreover, it was observed that when the cell strength equations were verified for a pillar with different element sizes (2.0, 2.5, 5.0, and 10.0 ft), the difference between the derived equations and the empirical strength equation increased and decreased for larger and smaller elements showing the effect of the element size on the pillar’s strength. Furthermore, the increment of the difference between the cell strength equations and the empirical equations is expected for larger elements, as observed in the calibration calculations for LaModel (Heasley et al., 2010). In this calibration, the difference between the cell strength equations and the empirical coal strength equation (Bieniawski, 1968) increments and decreases for larger or smaller elements, respectively.

Finally, the cell strength equations are implemented as an input parameter for the LaModel preprocessor LamPre for a case study mine in the Loyalhanna formation to obtain the stability analysis for the whole mine. It is observed that the derivation of the cell strength equations expands the usefulness and advantages of the S-Pillar strength equation (Esterhuizen et al., 2011). Moreover, with the integration of the cell strength equations into the BEM software LaModel (Heasley, 1998), the safety of the underground stone mine workers and the positive impact of the S-Pillar program can be improved further by a method that can generate the safety factor, and stress plots of a mine layout using the S-Pillar strength equation and the true overburden stress distribution. Additionally, this methodology allows the estimation of stress changes near the benched areas during the progressive benching stages in underground stone mines.
5.3. Suggestion for Future Studies

The safety factor plots with the true overburden stress distribution, generated by implementing the cell strength equations for stone mine pillars into the LaModel software, would also allow mine operators to identify the zones of potential pillar and roof stability hazards in their mine by overlaying their geological structure maps on such plots. Currently, S-Pillar calculates the impact of large angular discontinuities on the strength of a pillar layout through the large discontinuity factor (LDF) (Esterhuizen et al., 2011). Moreover, Esterhuizen et al. (2011) stated that if no large discontinuities are present, the LDF factor is equal to 1.0. Furthermore, this research used an LDF factor of 1.0. However, overlaying the map of the geological structures with the factor of safety and stress plots generated by LaModel, the LDF factor can be implemented into the cell strength equations derived in this research.

The strength of development and fully benched pillars is currently calculated using the empirical strength equation proposed by Esterhuizen et al. (2011). In 2006 Esterhuizen et al. presented the evaluation of the strength of slender pillars using the software FLAC3D developed by Itasca Consulting Group., where the authors implemented the bilinear constitutive model based on the Mohr-Coulomb strength criterion to simulate the brittle/frictional development of rock mass strength as a function of confining stress. Additionally, following a similar methodology, the pillar strength and stiffness variation behavior during the different stages of floor benching can be obtained and later can be implemented using the cell strength equations derived in this research.
REFERENCES


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