A study on supereulerian digraphs and spanning trails in digraphs

Omaema A. Lasfar
omaemalasfar@gmail.com

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A Study on Supereulerian Digraphs and Spanning Trails in Digraphs

Omaema Lasfar

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in
Mathematics

Hong-Jian Lai, Ph.D., Chair
John Goldwasser, Ph.D.
Rong Luo, Ph.D.
Charis Tsikkou Ph.D.
Brian D. Woerner Ph.D.

Department of Mathematics

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Abstract

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A strong digraph $D$ is eulerian if for any $v \in V(D)$, $d^+_D(v) = d^-_D(v)$. A digraph $D$ is supereulerian if $D$ contains a spanning eulerian subdigraph, or equivalently, a spanning closed directed trail. A digraph $D$ is trailable if $D$ has a spanning directed trail. This dissertation focuses on a study of trailable digraphs and supereulerian digraphs from the following aspects.

1. Strong Trail-Connected, Supereulerian and Trailable Digraphs.

For a digraph $D$, $D$ is trailable if $D$ has a spanning trail. A digraph $D$ is strongly trail-connected if for any two vertices $u$ and $v$ of $D$, $D$ posess both a spanning $(u,v)$-trail and a spanning $(v,u)$-trail. As the case when $u = v$ is possible, every strongly trail-connected digraph is also supereulerian. Let $D$ be a digraph. Let $S(D) = \{e \in A(D) : e$ is symmetric in $D\}$. A digraph $D$ is symmetric if $A(D) = S(D)$. The symmetric core of $D$, denoted by $J(D)$, has vertex set $V(D)$ and arc set $S(D)$. We have found a well-characterized digraph family $D$ each of whose members does not have a spanning trail with its underlying graph spanned by a $K_{2,n-2}$ such that for any strong digraph $D$ with its matching number $\alpha'(D)$ and arc-strong-connectivity $\lambda(D)$, if $n = |V(D)| \geq 3$ and $\lambda(D) \geq \alpha'(D) - 1$, then each of the following holds.

(i) There exists a family $\mathcal{D}$ of well-characterized digraphs such that for any digraph $D$ with $\alpha'(D) \leq 2$, $D$ has a spanning trial if and only if $D$ is not a member in $\mathcal{D}$.
(ii) If $\alpha'(D) \geq 3$, then $D$ has a spanning trail.
(iii) If $\alpha'(D) \geq 3$ and $n \geq 2\alpha'(D) + 3$, then $D$ is supereulerian.
(iv) If $\lambda(D) \geq \alpha'(D) \geq 4$ and $n \geq 2\alpha'(D) + 3$, then for any pair of vertices $u$ and $v$ of $D$, $D$ contains a spanning $(u,v)$-trail.

2. Supereulerian Digraph Strong Products.

A cycle vertex cover of a digraph $D$ is a collection of directed cycles in $D$ such that every vertex in $D$ lies in at least one cycle in this collection, and such that the union of the arc sets of these directed cycles induce a connected subdigraph of $D$. A subdigraph $F$ of a digraph $D$ is a circulation if for every vertex $v$ in $F$, the indegree of $v$ equals its outdegree, and a spanning circulation if $F$ is a cycle factor. Define $f(D)$ to be the smallest cardinality of a cycle vertex cover of the digraph $D/F$ obtained from $D$ by contracting all arcs in $F$, among all circulations $F$ of $D$. In [International Journal of Engineering Science Invention, 8 (2019) 12-19], it is proved that if $D_1$ and $D_2$ are nontrivial strong digraphs such that $D_1$ is supereulerian and $D_2$ has a cycle vertex cover $C'$ with $|C'| \leq |V(D_1)|$, then the Cartesian product $D_1$ and $D_2$ is also supereulerian. We prove that for strong digraphs $D_1$ and $D_2$, if for some cycle factor $F_1$ of $D_1$, the digraph formed from $D_2$ by contracting arcs in $F_1$ is hamiltonian with $f(D_2)$ not bigger than $|V(D_1)|$, then the strong product $D_1$ and $D_2$ is supereulerian.
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Chapter 1

1 Preliminary

1.1 Notations and Terminology

In this chapter, we will provide the common terminology and notation used in this dissertation.

We consider finite and simple graphs and digraphs. Undefined terms and notations will follow [15] for graphs and [9] for digraphs. Usually, we use $G$ to denote a graph and $D$ a digraph. Undefined terms and notations will follow [15] for graphs and [9] for digraphs. A directed graph (or just digraph) $D$ consists of a non-empty finite set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pairs of distinct vertices called arcs. We call $V(D)$ the vertex set and $A(D)$ the arc set of a digraph $D$. Throughout our discussions, we use the notation $(u, v)$ to denote an arc oriented from $u$ to $v$ in a digraph $D$; and use $[u, v]$ to denote either $(u, v)$ or $(v, u)$. When $[u, v] \in A(D)$, we say that $u$ and $v$ are adjacent. If two arcs of $D$ have a common vertex, we say that these two arcs are adjacent in $D$. If $(u, v)$ is an arc, we also say that $u$ dominates $v$ ($v$ is dominated by $u$). We say that a vertex $u$ is incident to an arc $e$ if $u$ is the head or tail of $e$. If $X$ is a vertex subset or an arc subset of $D$, we use $D[X]$ to denote the subdigraph of $D$ induced by $X$, $e(D)$ denotes the number of components of the underlying graph of $D$. If $e$ is an edge in a graph $G$ or an arc in a digraph $D$ incident with vertices $u$ and $v$, define $V(e) = \{u, v\}$. As in [9], we define, for a vertex $v \in V(D)$, $N^+_D(v) = \{w \in V(D) : (v, w) \in A(D)\}$, $N^-_D(v) = \{u \in V(D) : (u, v) \in A(D)\}$. The sets $N^+_D(v)$, $N^-_D(v)$ and $N_D(v) = N^+_D(v) \cup N^-_D(v)$ are called the out-neighbourhood, in-neighbourhood and neighbourhood of $v$. We call the vertices in $N^+_D(v)$, $N^-_D(v)$ and $N_D(v)$ the out-neighbours, in-neighbours and neighbours of $v$. For a subset $X \subseteq V(D)$, define $N_D(X) = \cup_{x \in X} N_D(x)$.

For an arc subset $F \subseteq A(D)$, define $V(F) = \cup_{e \in F} V(e)$ to be the set of vertices incident with an edge of $F$ in $D$. Following [9], for subsets $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\}, \text{ and } (X, Y)_G(D) = (X, Y)_D \cup (Y, X)_D.$$ 

If $X = \{x\}$ or $Y = \{y\}$, we often use $(x, Y)_D$ for $(X, Y)_D$ or $(X, y)_D$ for $(X, Y)_D$, respectively. Hence $(x, y)_D = (\{x\}, \{y\})_D$. For a vertex $v \in V(D)$, let $\partial^+_D(v) = (v, V(D) - v)_D$ and $\partial^-_D(v) = (V(D) - v, v)_D$. Thus $d^+_D(v) = |\partial^+_D(v)|$ and $d^-_D(v) = |\partial^-_D(v)|$. We further define $c(D)$ denotes the number of components of the underlying graph of $D$. In addition, we define the minimum out-degree (minimum in-degree, respectively) of $D$ to be

$$\delta^+(D) = \min\{d^+_D(v) : v \in V(D)\}, \delta^-(D) = \min\{d^-_D(v) : v \in V(D)\}, \text{ respectively}.$$ 

Following [15], $\kappa(G)$, $\kappa'(G)$ and $\alpha(G)$ denote the connectivity, the edge connectivity and the independence number of a graph $G$; and $\kappa(D)$ and $\lambda(D)$ denotes the vertex-strong connectivity and the arc-strong connectivity of a digraph $D$, respectively. If $D$ is a digraph, we often use $G(D)$ to denote the underlying undirected graph of $D$, the graph obtained from $D$ by erasing all orientation on the arcs of $D$. The stability number $\alpha(D)$, and the matching number $\alpha'(D)$, of a digraph $D$ are defined
\[ \alpha(D) = \alpha(G(D)) \text{ and } \alpha'(D) = \alpha'(G(D)), \]

By the definition of \( \lambda(D) \) in [9], we note that for any integer \( k \geq 0 \) and a digraph \( D \),

\[ \lambda(D) \geq k \text{ if and only if for any nonempty proper subset } X \subset V(D), |\partial_P^+(X)| \geq k. \]

We use paths, cycles, and trails as defined in [15] when the discussion is on an undirected graph \( G \), and to denote directed paths, directed cycles and directed trails when the discussion is on a digraph \( D \). A directed trail (or path, respectively) from a vertex \( u \) to a vertex \( v \) in a digraph \( D \) is often refereed as to a \((u,v)\)-trail (a \((u,v)\)-path, respectively). For an integer \( n \), we define \([n] = \{1, 2, \ldots, n\}\). A walk in \( D \) is an alternating sequence \( W = x_1a_1x_2a_2x_3 \cdots x_{k-1}a_{k-1}x_k \) of vertices \( x_i \) and arcs \( a_j \) from \( D \) such that \( a_j = (x_j, x_{j+1}) \) for every \( i \in [k] \) and \( j \in [k-1] \). A walk \( W \) is closed if \( x_1 = x_k \), and open otherwise. We use \( V(W) = \{x_i : i \in [k]\} \) and \( A(W) = \{a_j : j \in [k-1]\} \). We say that \( W \) is a walk from \( x_1 \) to \( x_k \) or an \((x_1, x_k)\)-walk. If \( x_1 \neq x_k \), then we say that the vertex \( x_1 \) is the initial vertex of \( W \), the vertex \( x_k \) is the terminal vertex of \( W \), and \( x_1 \) and \( x_k \) are end-vertices of \( W \). The length of a walk is the number of its arcs. When the arcs of \( W \) are understood from the context, we will denote \( W \) by \( x_1x_2 \cdots x_k \). A ditrail in \( D \) is a walk in which all arcs are distinct. A ditrail is often considered as a subdigraph induced by the arcs in the trail. If the vertices of \( W \) are distinct, then \( W \) is a dipath. If the vertices in the trail \( x_1x_2 \cdots x_{k-1} \) are distinct, then \( W \) is a dicycle. We say that an ordered pair of vertices \((x, y)\) is dominated (dominating, respectively) if there exists \( z \in V(D) \), with \((z, x), (z, y) \in A(D)\) or \((x, z), (y, z) \in A(D)\), respectively).

An Eulerian trail (or Eulerian tour) of \( G \) is a trail in \( G \) that visits every edge exactly once (allowing for revisiting vertices). For a graph \( G \), denote \( O(G) = \{v \in V(G) : d_G(v) \text{ is odd}\} \). A graph with \( O(G) = \emptyset \) is called an even graph.

**Theorem 1.1** (Euler, 1736) The following are equivalent for a graph \( G \).
(i) \( G \) contains an Euler tour.
(ii) \( G \) is connected and \( O(G) = \emptyset \).

A graph \( G \) is eulerian if \( G \) is a connected with \( O(G) = \emptyset \). A graph \( G \) is supereulerian if \( G \) has a spanning eulerian subgraph. Thus a graph \( G \) is supereulerian if \( G \) has a spanning closed trail. The supereulerian graph problem, raised by Boesch, Suffel, and Tindell [16], seeks to characterize supereulerian graphs. Pulleyblank [43] showed that determining whether a graph is supereulerian, even when restricted to planar graphs, is \( \mathcal{NP} \)-complete. For more literature on supereulerian graphs, see Catlin’s survey [17] and its supplement by Z.Chen et.al. [20] and the updating in [34]. The supereulerian graph problem is also motivated by the study of hamiltonian problems of graphs. A graph \( G \) is hamiltonian if \( G \) has a spanning cycle.

A walk (path, cycle) \( W \) is a Hamilton (or hamiltonian) walk (path, cycle) if \( V(W) = V(D) \). A digraph \( D \) is hamiltonian if \( D \) contains a Hamilton cycle. A trail \( W = x_1x_2 \cdots x_k \) is an Euler (or eulerian) trail if \( A(W) = A(D) \), \( V(W) = V(D) \) and \( x_1 = x_k \). For digraphs, a strong digraph \( D \) is
**Definition 1.3** \cite{37} For a digraph $D$, an arc $[u,v] \in A(D)$ is a **symmetric** in $D$ if both arcs $(u,v)$ and $(v,u)$ are adjacent, $v \in V(D)$, $d^+_H(v) = d^-_H(v)$. The following is well-known or immediately from the definition.

**Theorem 1.2** (Euler, see Theorem 1.7.2 of \cite{9} and Veblen \cite{46}) Let $D$ be a digraph. The following are equivalent.

(i) $D$ is eulerian.
(ii) $D$ is a spanning closed trail.
(iii) $D$ is a disjoint union of dicycles and $D$ is connected.

The supereulerian problem in digraphs was considered by Gutin \cite{25}. A digraph $D$ is supereulerian if $D$ contains a spanning eulerian subdigraph, or equivalently, a spanning closed trail. Thus supereulerian digraphs must be strong, and every hamiltonian digraph is also a supereulerian digraph.

A digraph $D$ is **trialable** if there exist $u,v \in (D)$, such that $D$ has a spanning $(u,v)$-trail. A digraph $D$ is a **strong** if, for every pair $u,v$ of distinct vertices in $D$, there exists an $(u,v)$-walk; and $D$ is a **weakly connected** if $G(D)$ is a connected.

A digraph $D$ is **strongly trail-connected** if for any two vertices $u$ and $v$ of $D$, $D$ posses both a spanning $(u,v)$-trail and a spanning $(v,u)$-trail. As the case when $u = v$ is possible, every strongly trail-connected digraph is also supereulerian.

Given a digraph $D$, we define the **path covering number** of $D$, $pc(D)$, as the minimum possible number of vertex-disjoint paths covering the vertices of $D$ and the **trail covering number** of $D$, $\tau(D)$, as the minimum possible number of arc-disjoint trails covering the vertices of $D$. Note that some of these trails may consist of a single vertex.

A graph $G$ is complete, if every pair of distinct vertices in $G$ are adjacent. We will denote the complete graph on $n$ vertices (which is unique up to isomorphism) by $K_n$. Its complement $K_n^*$ has no edge. A digraph $D$ is complete if, for every pair $u,v$ of distinct vertices of $D$, both $(u,v)$ and $(v,u)$ are in $D$. The complete digraph on $n$ vertices will be denoted by $K_n^*$.

Let $e = [v_1,v_2] \in A(D)$ be an arc of $D$. Define $D/e$ to be the digraph obtained from $D - e$ by identifying $v_1$ and $v_2$ into a new vertex $v_e$, and deleting the possible resulting loop(s). If $W \subseteq A(D)$ is an arc subset, then define the **contraction** $D/W$ to be the digraph obtained from $D$ by contracting each arc $e \in W$, and deleting any resulting loops. Thus even $D$ does not have parallel arcs, a contraction $D/W$ is loopless but may have parallel arcs. If $H$ is a subdigraph of $D$, then we often use $D/H$ for $D/A(H)$. If $L$ is a connected component of $H$ and $v_L$ is the vertex in $D/H$ onto which $L$ is contracted, then $D[V(L)]$ is the **contraction preimage** of $v_L$. We adopt the convention to define $D/\emptyset = D$, and define a vertex $v \in V(D/W)$ to be a **trivial vertex** if the preimage of $v$ is a single vertex (also denoted by $v$) in $D$. Hence we often view trivial vertices in a contraction $D/W$ as vertices in $D$.

For a graph $G$, a **matching** $M$ of $G$ is a subset of edges of $G$ its elements are links and no two are adjacent in $G$. Let $M$ be a matching in a graph $G$. A path $P$ is an $M$–**augmenting path**, if the edges of $P$ are alternately in $M$ and in $E(G) - M$, and if both end vertices of $P$ are not in $V(M)$. An $M$–augmenting path of a digraph $D$ is an $M$–augmenting path of $G(D)$.
and \((v, u)\) are in \(A(D)\). In particular, a symmetric dipath \(P\) is a dipath such that every arc of \(P\) is symmetric.

**Definition 1.4** [4] Let \(D\) be a digraph such that either \(D = K_1\) or \(A(D) \neq \emptyset\). If for any \(u, v \in V(D)\), \(D\) contains a symmetric dipath from \(u\) to \(v\), then \(D\) is called a symmetrically connected digraph.

**Definition 1.5** [4] Let \(c \geq 2\) be an integer and let \(D\) be a weakly connected digraph and let \(\{H_1, H_2, \ldots, H_c\}\) be the set of maximal symmetrically connected subdigraphs of \(D\). If for any proper nonempty subset \(J \subset \{H_1, H_2, \ldots, H_c\}\), there exist an \(H_i \in J\) and a vertex \(v \in V(H_i)\), and an \(H_j \notin J\) such that \(N^+_D(v) \cap V(H_j) \neq \emptyset\) and \(N^-_D(v) \cap V(H_j) \neq \emptyset\), then \(D\) is a partially symmetric.

A digraph \(D = (V, A)\) is a **semicomplete** if \(D\) is without nonadjacent vertices. Bang-Jenson and Gutin in [9] defined a locally semicomplete digraph as following, a digraph \(D\) is a **locally in- semicomplete (locally out-semicomplete)** if for every vertex \(x\) of \(D\), the in-neighbours (out-neighbours) of \(x\) induce a **semicomplete** digraph. A digraph \(D\) is **locally semicomplete** if it is both locally in-semicomplete and locally out-semicomplete.

A digraph \(D = (V, A)\) is a **semicomplete multipartite** if there is a partition \(V_1, V_2, \ldots, V_c\) of \(V\) into independent sets so that every vertex in \(V_i\) shares an arc with every vertex in \(V_j\) for \(1 \leq i < j \leq c\).

**Definition 1.6** [8] A **locally semicomplete multipartite** digraph \(D\) is obtained from a locally semicomplete digraph \(F\) with \(V(F) = \{v_1, v_2, \ldots, v_q\}\) by blowing up each vertex \(v_i \in V(F)\) into one independent set \(V_i\) in \(D\), such that \(N^\lambda_D(x) = V_i_1 \cup \cdots \cup V_i_p\) for any \(x \in V_i\) if and only if \(N^\lambda_F(v_i) = \{v_{i_1}, \ldots, v_{i_p}\}\), where \(\lambda \in \{+, -\}\) and \(\{v_{i_1} \cup \cdots \cup v_{i_p}\} \subset V(F)\).

**Definition 1.7** [9] A digraph \(D\) is **transitive**, if for every pair \((x, y)\) and \((y, z)\) of arcs in \(D\) with \(x \neq z\), the arc \((x, z)\) is also in \(D\). A digraph \(D\) is a **quasi-transitive**, if for every triple \(x, y, z\) of distinct vertices of \(D\) such that \((x, y)\) and \((y, z)\) are arcs of \(D\), there is at least one arc between \(x\) and \(z\). Clearly, a semicomplete digraph is a quasi-transitive.

The following theorem is an equivalent definition of a strong quasi-transitive digraph.

**Theorem 1.8** (Canonical Decomposition, Bang-Jenson and Huang, Theorem 3.5 of [13]) Let \(D\) be a strong quasi-transitive digraph, then there exist a strong semicomplete digraph \(S\) on \(s\) vertices and quasi-transitive digraphs \(Q_1, \ldots, Q_s\) such that \(D = S|Q_1, \ldots, Q_s\).

For an integer \(k \geq 2\), let \(P_k\) denote the dipath on \(k\) vertices. A subdigraph \(H\) of a digraph \(D\) is a \(P_k\)–subdigraph if \(H \cong P_k\). If \(D\) does not have an induced \(P_k\), then for any \(P_k\)-subdigraph \(H\) of \(D\), we must have \(|A(D[V(H)])| \geq k\).

**Definition 1.9** [5] For integers \(h \geq k \geq 2\), defined \(\mathcal{F}(P_k, h)\) to be the family of all simple digraphs such that \(D \in \mathcal{F}(P_k, h)\) if and only if \(D\) is strong and satisfies both of the following.

(i) \(D\) contains at least one dipath \(P_k\) with \(|A(D[V(P_k)])| = h\), and
(ii) for any dipath \(P_k\) in \(D\), \(|A(D[V(P_k)])| \geq h\).
A graph G to be locally connected, if for every vertex \( v \in V(G) \), the vertices adjacent to \( v \) induce a connected subgraph in \( G \). M. Algefari et al [3], defined the following.

**Definition 1.10** [3] For a vertex \( v \in V(D) \) is \( k^+ \)-locally-arc-connected, (or \( k^- \)-locally-arc-connected, or \( k \)-locally-arc-connected, respectively) if \( \lambda(D[N^+(v)]) \geq k(\lambda(D[N^-(v)]) \geq k, \) or \( \lambda(D[N(v)]) \geq k, \) respectively). A digraph \( D \) is \( k^+ \)-locally-arc-connected, (or \( k^- \)-locally-arc-connected, or \( k \)-locally-arc-connected, respectively) if every vertex of \( D \) is \( k^+ \)-locally-arc-connected, (or \( k^- \)-locally-arc-connected, or \( k \)-locally-arc-connected, respectively).

**Definition 1.11** [24] For any distinct four vertices \( c_1, c_2, c_3, c_4 \) of \( D \), \( D \) is \( \mathcal{H}_1 \)-quasi-transitive if \( c_1 \rightarrow c_2 \leftarrow c_3 \leftarrow c_4, c_1 \) and \( c_4 \) are adjacent; \( D \) is \( \mathcal{H}_2 \)-quasi-transitive if \( c_1 \leftarrow c_2 \rightarrow c_3 \rightarrow c_4, c_1 \) and \( c_4 \) are adjacent; \( D \) is \( \mathcal{H}_3 \)-quasi-transitive if \( c_1 \rightarrow c_2 \rightarrow c_3 \rightarrow c_4, c_1 \) and \( c_4 \) are adjacent; \( D \) is \( \mathcal{H}_4 \)-quasi-transitive if \( c_1 \rightarrow c_2 \leftarrow c_3 \rightarrow c_4, c_1 \) and \( c_4 \) are adjacent. There are four distinct possible orientations of a 3-path; therefore, \( \mathcal{H}_i \)-quasi-transitive digraphs as \( 3 \)-path-quasi-transitive digraphs for convenience, where \( i \in \{1, 2, 3, 4\} \).

**Definition 1.12** [6] Let \( D \) be a digraph, \( C_1, C_2, \ldots, C_k \) be cycle subdigraphs of \( D \) and set \( \mathcal{F} = \{C_1, C_2, \ldots, C_k\} \), where \( k > 0 \) is an integer. \( \mathcal{F} \) is called an cycle vertex cover of \( D \), if both

(i) \( V(D) = \bigcup_{C_i \in \mathcal{F}} V(C_i) \); and

(ii) \( \bigcup_{C_i \in \mathcal{F}} C_i \) is weakly connected.

**Definition 1.13** [36] Let \( D \) be a digraph. We define \( D \) to be a circulation if for any \( v \in V(D) \), we have \( d_D^+(v) = d_D^-(v) > 0 \); and \( D \) is eulerian if \( D \) is a spanning connected circulation. A subdigraph \( F \) of \( D \) is a cycle factor if \( F \) is a spanning circulation, or equivalently, \( F \) is a collection of arc-disjoint cycles spanning \( V(D) \).

By definition, if \( D \) is a circulation, then every component of \( D \) is eulerian. By Theorem 1.2, we observe the following

Every circulation is an arc-disjoint union of cycles. \hspace{1cm} (1)

Thus, for a subdigraph \( F \) of \( D \) is a cycle factor, if \( F \) is a collection of arc-disjoint cycles spanning \( V(D) \).

**Definition 1.14** [36] Let \( F \) be a circulation of a digraph \( D \) and \( D/F \) denote the digraph formed from \( D \) by contracting arcs in \( A(F) \), for any circulation \( F \) of \( D \), define

(i) \( f_D(F) = \min\{|C| : C \) is a cycle vertex cover of \( D/F\} \) and,

(ii) \( f(D) = \min\{f_D(F) : F \) is a circulation of \( D\} \).

The following is well-known or immediately from the definition. Following [29], some digraph products are defined as follows.
Definition 1.15 [29] Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs, such that

$$V_1 = \{u_1, u_2, \ldots, u_{n_1}\} \text{ and } V_2 = \{v_1, v_2, \ldots, v_{n_2}\}$$

Then the Cartesian product, the Direct product and the Strong product of $D_1$ and $D_2$ are defined as following,

(i) **The Cartesian product** denoted by $D_1 \square D_2$ is the digraph with vertex set $V_1 \times V_2$ and

$$A(D_1 \square D_2) = \{(u_i, v_j) : u_i = u_s \text{ and } v_j = v_t \in A_2, \text{ or } u_i u_s \in A_1 \text{ and } v_j = v_t \}.$$

(ii) **The Direct product** denoted by $D_1 \times D_2$ is the digraph with vertex set $V_1 \times V_2$ and

$$A(D_1 \times D_2) = \{(u_i, v_j) : u_i u_s \in A_1 \text{ and } v_j v_t \in A_2 \}.$$

(iii) **The Strong product** denoted by $D_1 \otimes D_2$ is the digraph with vertex set $V_1 \times V_2$ and

$$A(D_1 \otimes D_2) = \{(u_i, v_j) : u_i = u_s \text{ and } v_j v_t \in A_2, \text{ or } u_i u_s \in A_1 \text{ and } v_j = v_t \text{ or both } u_i u_s \in A_1 \text{ and } v_j v_t \in A_2 \}.$$

v) **The Lexicographic product** denoted by $D_1 [D_2]$ is the digraph with vertex set $V_1 \times V_2$ and

$$A(D_1 [D_2]) = \{((u_i, v_j), (u_s, v_t)) : u_i = u_s \text{ and } (v_j, v_t) \in A_2 \text{ or } (u_i, u_s) \in A_1 \}.$$

The following figures illustrate the definition of the Cartesian product (Fig. 1.), the Direct product (Fig. 2.) and Strong product (Fig. 3.) of $P_4$ and $C_3$.

Figure 1. The digraphs $P_4$, $C_3$ and the Cartesian product $P_4 \square C_3$
Figure 2. The digraphs $P_4$, $C_3$ and the Direct product $P_4 \times C_3$

Figure 3. The digraphs $P_4$, $C_3$ and the Strong product $P_4 \boxtimes C_3$
1.2 Main Results

This dissertation focuses on a study of dicycle cover and supereulerian digraphs from the following aspects.

1. **Strong trail-connected, Supereulerian and Tirable Digraphs.** Digraphs.

For a digraph $D$, $D$ is tirable digraph if $D$ has a spanning trail. A digraph $D$ is strongly trail-connected if for any two vertices $u$ and $v$ of $D$, $D$ possesses both a spanning $(u,v)$-trail and a spanning $(v,u)$-trail. As the case when $u = v$ is possible, every strongly trail-connected digraph is also supereulerian. Let $D$ be a digraph. Let $S(D) = \{e \in A(D) : e$ is symmetric in $D\}$. A digraph $D$ is symmetric if $A(D) = S(D)$. The symmetric core of $D$, denoted by $J(D)$, has vertex set $V(D)$ and arc set $S(D)$. We have found a well-characterized digraph family $D$ each of whose members does not have a spanning trail with its underlying graph spanned by a $K_{2,n-2}$ such that for any strong digraph $D$ with matching number $\alpha'(D)$ and arc-strong-connectivity $\lambda(D)$, if $n = |V(D)| \geq 3$ and $\lambda(D) \geq \alpha'(D) - 1$, then each of the following holds.

(i) There exists a family $D$ of well-characterized digraphs such that for any digraph $D$ with $\alpha'(D) \leq 2$, $D$ has a spanning trial if and only if $D$ is not a member in $D$.
(ii) If $\alpha'(D) \geq 3$, then $D$ has a spanning trail.
(iii) If $\alpha'(D) \geq 3$ and $n \geq 2\alpha'(D) + 3$, then $D$ is supereulerian.
(iv) If $\lambda(D) \geq \alpha'(D) \geq 4$ and $n \geq 2\alpha'(D) + 3$, then for any pair of vertices $u$ and $v$ of $D$, $D$ contains a spanning $(u,v)$-trail.

2. **Supereulerian Digraph Strong Products.** A cycle vertex cover of a digraph $D$ is a collection of directed cycles in $D$ such that every vertex in $D$ lies in at least one dicycle in this collection, and such that the union of the arc sets of these directed cycles induce a connected subdigraph of $D$. A subdigraph $F$ of a digraph $D$ is a circulation if for every vertex $v$ in $F$, the indegree of $v$ equals its outdegree, and a spanning circulation if $F$ is a cycle factor. Define $f(D)$ to be the smallest cardinality of a cycle vertex cover of the digraph $D/F$ obtained from $D$ by contracting all arcs in $F$, among all circulations $F$ of $D$. In [International Journal of Engineering Science Invention, 8 (2019) 12-19], it is proved that if $D_1$ and $D_2$ are nontrivial strong digraphs such that $D_1$ is supereulerian and $D_2$ has a cycle vertex cover $C'$ with $|C'| \leq |V(D_1)|$, then the Cartesian product $D_1$ and $D_2$ is also supereulerian. We prove that for strong digraphs $D_1$ and $D_2$, if for some cycle factor $F_1$ of $D_1$, the digraph formed from $D_1$ by contracting arcs in $F_1$ is hamiltonian with $f(D_2)$ not bigger than $|V(D_1)|$, then the strong product $D_1$ and $D_2$ is supereulerian.
Chapter 2

2 Literature Review

2.1 Related Results in undirected Graphs

In this section, we will give a brief review of supereulerian undirected graphs. In 1962, a Chinese mathematician called Kuan Mei-Ko was interested in a postman delivering mail to a number of streets such that the total distance walked by the postman was as short as possible. Motivated by the Chinese Postman Problem, Boesch et al. [16] proposed the supereulerian problem which determines if a graph has a spanning eulerian subgraph. They indicated that this might be a difficult problem. Pulleyblank [43] showed that such a decision problem, even when restricted to planar graphs, is \( \mathcal{NP} \)-complete. Since then, there have been lots of researches on this topic. Catlin [17] in 1992 presented the first survey on supereulerian graphs. Later Chen et al. [20] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A latest survey on supereulerian graphs is given in [34].

The following corollary provides a sufficient condition for the existence of edge-disjoint spanning trees of cardinality \( k \).

**Corollary 2.1** ([42], [33], [28]) Every finite \( 2k \)-edge-connected graph has \( k \) edge-disjoint spanning trees.

Jaeger [32] and Catlin [18] independently showed the following theorem.

**Theorem 2.2** (Jeager [32], Catlin [18]) Every 4-edge-connected graph is supereulerian.

**Theorem 2.3** (Catlin, Corollary 1 of [18]) There exist graph families \( \mathcal{F} \) such that if every edge of a connected graph \( G \) lies in a subgraph of \( G \) isomorphic to a member in \( \mathcal{F} \), then \( G \) is supereulerian. In particular, if every edge of \( G \) lies in a 3-cycle of \( G \), then \( G \) is supereulerian.

For \( X \subset E(G) \), the contraction \( G/X \) is obtained from \( G \) by contracting each edge of \( X \) and deleting the resulting loops. If \( H \subset G \), we write \( G/H \) for \( G/E(H) \). If \( H \) is connected, let \( v_H \) denote the vertex in \( G/H \) to which \( H \) is contracted, in this case, \( H \) is called the preimage of \( v_H \). A graph \( G \) is a **collapsible** [18], if for every even subset \( R \subset V(G) \), \( G \) has a spanning connected subgraph \( H_R \) of \( G \) with \( O(H_R) = R \). In particular, \( K_1 \) is both supereulerian and collapsible and any collapsible graph \( G \) is supereulerian. In [18], Catlin showed that every graph \( G \) has a unique collection of pairwise disjoint maximal collapsible subgraphs \( H_1, H_2, ..., H_c \). The graph obtained from \( G \) by contracting each \( H_i \) into a single vertex \( 1 \leq i \leq c \), is called the **reduction** of \( G \). A graph is reduced if it is the reduction of some other graph. For undirected graph \( G \), Catlin [18] proved that if \( G \) has two edge-disjoint spanning tree, then \( G \) is collapsible which implies that \( G \) is supereulerian. Earlier, Jaeger in [32] proved that such graphs must be supereulerian.

**Theorem 2.4** [32] If a graph \( G \) has two edge-disjoint spanning trees, then \( G \) is supereulerian.
Catlin, in [18], showed the following theorem.

**Theorem 2.5** (Catlin’s Reduction Method)[18] Let $G$ be a connected graph and $G'$ be the reduction of $G$. Let $H$ be a collapsible subgraph of $G$. Then each of the following holds.

(i) $G$ is collapsible if and only if $G/H$ is collapsible. In particular, $G$ is collapsible if and only if $G' = K_1$.

(ii) $G$ is supereulerian if and only if $G/H$ is supereulerian. In particular, $G$ is supereulerian if and only if $G'$ is supereulerian.

Let $F(G)$ denote the minimum number of edges that must be added to $G$ in order to obtain a graph that has two edge-disjoint spanning trees. Thus, Theorem 2.4 says that if $F(G) = 0$, then $G$ is supereulerian. Catlin [18] defined the reduction of a graph.

**Theorem 2.6** (Theorem 7 of Catlin [18]). If $F(G) \leq 1$; then either $G$ is supereulerian or $G$ can be contracted to $K_2$.

**Theorem 2.7** (Theorem 1.5 of Catlin et al. [19]). Let $G$ be a connected graph. If $F(G) \leq 2$, then exactly one of the following holds

(i) $G$ is supereulerian;

(ii) $G$ has a cut-edge(bridge);

(iii) The reduction of $G$ is $K_{2,s}$ for some odd integer $s \geq 3$.

Motivated by the above result, H-J. Lai and H. Yan [35] obtained the following result for 2-edge-connected simple graphs.

**Theorem 2.8** (Lai and Yan, Theorem 2 of [35]) If $G$ is a 2-edge-connected simple graph and $\alpha'(G) \leq 2$, then $G$ is supereulerian if and only if $G$ is not $K_{2,t}$ for some odd number $t$.

**2.2 Necessary Condition for Supereulerian Digraphs**

In this section, we introduce necessary conditions to a digraph to be supereulerian. The first necessary condition for a digraph to be supereulerian is presented by Y. Hong et al. [30]. In [30] they introduced the following definition.

**Definition 2.9** [30] Let $D$ be a strong digraph and $U \subset V(D)$. Then $D[U]$, the digraph induced by $U$, has ditrails $P_1, \ldots, P_t$ such that

(i) $\bigcup_{i=1}^t V(P_i) = U$; and

(ii) $A(P_i) \cap A(P_j) = \emptyset$ for any $i \neq j$.

Let $\tau(U)$ be the minimum value of such $t$. Then $c(G(D[U])) \leq \tau(U) \leq |U|$ where $c(G(D[U]))$ is the number of components of the underlying graph of $D[U]$.

For any $A \subset V(D) - U$, denote $B := V(D) - U - A$ and let

$h(U, A) := \min\{\delta_D^+(A), \delta_D^-(A)\} + \min\{|(U, B)_D, |(B, U)_D|\} - \tau(U)$, and

$h(U) := \min\{h(U, A) : A \cap U = \emptyset\}$. 


The next proposition has been provided by Y. Hong et al. [30] as a necessary condition for a digraph $D$ to be supereulerian. It has been used to show that there exists a families of strong digraphs each of which contains no spanning eulerian subdigraphs (non-supereulerian).

**Proposition 2.10** (Hong, Lai and Liu, Proposition 2.1 of [30]) If a digraph $D$ has a spanning eulerian subdigraph, then for any $U \subset V(D)$, $h(U) \geq 0$.

In the rest, we will display some of the results that have used Proposition 2.10 to construct the infinity families of non-supereulerian digraphs.

**Example 2.11** [30] Let $k_1, k_2, l \geq 2$ be integers, and $D_1$ and $D_2$ be two disjoint complete digraphs of order $k_1 + 1$ and $k_2 + 1$, respectively, and let $U$ be an independent set of size $\ell$ such that $(V(D_1) \cup V(D_2)) \cap U = \emptyset$. Let $D(k_1, k_2, \ell)$ denote the family of digraphs such that $D \in D(k_1, k_2, \ell)$ if and only if $D$ is the digraph obtained from $D_1 \cup D_2 \cup U$ by adding all arcs directed from every vertex in $U$ and $D_2$ to every vertex in $D_1$, and all arcs directed from every vertex in $D_2$ to every vertex in $U$, and then by adding an set of $l - 1$ arcs directed from some vertices in $D_1$ to some vertices in $D_2$. Assume $k_1, k_2 \geq \ell - 1$. For any $D \in D(k_1, k_2, \ell)$, $V(D) = k_1 + k_2 + \ell + 2$, and $D$ is a strong digraph with minimum degree $\delta^+(D) = k_1$ and $\delta^-(D) = k_2$. Let $A = V(D_1)$. Then

$$h(U, A) = |\partial_D^+(A)| + |(U, V(D) - U - A)_D| - \tau(U) = (\ell - 1) - \ell < 0.$$ 

By Proposition 2.10, $D$ does not have a spanning eulerian subdigraph.

**Example 2.12** [31] Let $k_1, k_2 \geq 2$ be integers and for any $i \in \{1, 2\}$. Let $D(i, k_2, 3)$ and $D(k_1, i, 3)$ be families of digraphs defined as Example 2.11. Let $D_2 \subset \bigcup_{i=1}^{2}(D(i, k_2, 3) \cup D(k_1, i, 3))$ be the family of digraphs with $\delta^+(D) = \delta^-(D) = 2$ for each $D \in D_2$. As each $D \in \bigcup_{i=1}^{2}(D(i, k_2, 3) \cup D(k_1, i, 3))$, $D \in D(k_1, k_2, \ell)$. By Example 2.11, $D$ contains no spanning closed ditrails. Thus, every one in $D_2$ is non-supereulerian.

**Example 2.13** [31] Let $k_1, k_2 \geq 2$ be integers, let $D(0, k_2, 2)$ and $D(k_1, 0, 2)$ be infinity families defined as Example 2.11 where $U = \{u_1, u_2\}$. Let $D$ be the set of digraphs obtained from digraphs in $D(0, k_2, 2) \cup D(k_1, 0, 2)$ by replacing a vertex in $U$ by a dicycle $u_1u_2u_1$ of length 2 and adding all the arcs from $\{u_1, u_2\}$ to $V(D_1)$ and all the arcs from $V(D_2)$ to $\{u_1, u_2\}$. Let $D \in D_3$, let $A = V(D_1)$ and $V(D) - U - A = V(D_2)$. As $\tau(U) = 2$, then

$$h(U, A) = \min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + \min\{|(U, V(D) - U - A)_D|, |(V(D) - U - A, U)_D|\} - \tau(U) = 1 + 0 - 2 < 0.$$ 

Thus, $D$ is non-supereulerian by Proposition 2.10.

**Example 2.14** [4] Let $\alpha, \beta, k > 0$ be integers with $\alpha, \beta \geq k + 1$, and let $A$ and $B$ be two disjoint set of vertices with $|A| = \alpha$ and $|B| = \beta$ . Let $l \geq \alpha \beta + 1$ be an integer and let $U$ be an independent set of size $\ell$ such that $(A \cup B) \cap U = \emptyset$. Let $D = D(\alpha, \beta, k, \ell)$ is a digraph obtained from $V(D) = A \cup B \cup U$ by adding all arcs directed from every vertex in $U$ and in $B$ to every vertex in $A$ and all arcs directed from
every vertex in $B$ to every vertex in $U$, and then by adding all arcs directed from every vertex in $A$ to
every vertex in $B$. (See Fig. 4.) Thus $D[A \cup B] \cong K_{\alpha+\beta}^*$ and for any $u \in U$, $N^+_D(u) = A$, $N^-_D(u) = B$. 
As $|\partial^+_D(A)| = \alpha$, and $|(U, B)_D| = 0$ and so $\tau(U) = |U| > \alpha \beta$. Therefore we have

$$h(U, A) = |\partial^+_D(A)| + |(U, B)_D| - \tau(U) = \alpha \beta - |U| < 0.$$ 

It follows from Proposition 2.10, $D$ is non-supereulerian.

![Figure 4. The digraph $D = D(\alpha, \beta, k, \ell)$.

From Example 2.14, M. Algefari et al. [4] showed that there exists an infinite family of non-supereulerian digraphs with arbitrarily high arc-strong connectivity such that every arc of each of these digraphs lies in a directed 3-cycle. Hence both Theorem 2.2 and Theorem 2.3 cannot be directly extended to digraphs. Moreover, it follows from Definition 1.9 that the previous example investigated forbidden induced subdigraph conditions to assure the existence of non-supereulerian digraphs where Algefari et al. in [5] proved that digraphs in $\mathcal{F}(P_3, h)$ with $3 \leq h \leq 4$ are not necessarily super eulerian, as can be seen in the Example 2.14 above. Since any $D \in D(\alpha, \beta, k, \ell)$ is non-supereulerian. By Definition 1.9, $D \in \mathcal{F}(P_3, 4)$.

The $k$-locally-arc-connected digraphs are defined at Definition 1.10, M. Algefari, H-J. Lai, J. Xu [3] showed that Proposition 2.10 can be applied to show that there exists a family of strong and locally $k^+$-arc-connected which is non-supereulerian digraphs and non-supereulerian locally $k$-arc-connected digraphs. The following have been proved by Algefari et al. [3] to show that the following digraph $D = D(n_1, n_2, \ell) \in \mathcal{D}(k, \ell)$ is a locally $k^+$-arc-connected digraph that is non-supereulerian, also they proved that $D$ is a locally $k$-arc-connected digraph.

**Example 2.15** [3] Let $k > 0$, $\ell > (k + 1)^2$ and $n_1 \geq n_2 \geq k + 2$ be integers, $D_1$ and $D_2$ be two vertex disjoint complete digraphs on $n_1$ and $n_2$ vertices, respectively, $X \subset V(D_1)$ and $Y \subset V(D_2)$ with $|X| = |Y| = k + 1$ and let $U$ be a set of independent vertices of size $\ell$ such that $(V(D_1) \cup V(D_2)) \cap U = \emptyset$. Let $\mathcal{D}(k, \ell)$ denote the family of digraphs such that $D = D(n_1, n_2, \ell) \in \mathcal{D}(k, \ell)$ if and only if $D$ is the digraph obtained from the disjoint union $D_1 \cup D_2 \cup U$ by adding all arcs directed from every vertex in $U$ and $D_2$ to every vertex in $D_1$, and all arcs directed from every vertex in $D_2$ to every vertex in $U$, and then by adding $(k + 1)^2$ arcs from $X$ to $Y$. (See Fig. 5.) In [3], they proved that $D$ is a locally $k^+$-arc-connected digraph. By applying Proposition 2.10, Let $A = V(D_1)$. Then
\[ h(U, A) = \left| \partial_D^+(A) \right| + \left| (U, V(D) - U - A)_D \right| - \tau(U) = (k + 1)^2 + 0 - \ell < 0. \]

Thus, \( D \) is non-supereulerian.

![Figure 5](image)

Figure 5. The digraph \( D = D(n_1, n_2, \ell) \), with \( n_1, n_2 \geq k + 2 \), and \( \ell > (k + 1)^2 \).

The following example indicate that there exists a family of non-supereulerian bipartite digraphs.

**Example 2.16** [48] Let \( k > 0 \) and \( \ell \geq \left\lceil \frac{k}{2} \right\rceil 2 + 1 \) be integers, \( a, b \) be even integers with \( a \leq b \) and \( a + b = 2k \), and let \( A \) and \( B \) be two disjoint sets of vertices with \( |A| = a \) and \( |B| = b \). Let \( U \) be an independent set of size \( \ell \) such that \((A \cup B) \cap U = \emptyset\). Define a digraph \( D = D(a, b, k, \ell) \) such that \( V(D) = A \cup B \cup U \) and \( A(D) \) consists exactly the arcs satisfying the following (See Fig. 6).

\[ \begin{align*}
(D1) & \quad D[A] \text{ is a complete bipartite digraph } k\left(\frac{a}{2}, \frac{b}{2}\right) \text{ with vertex bipartition } (X_1, Y_1) \text{ such that } |X_1| = |Y_1| = \frac{a}{2}; \text{ and } D[B] \text{ is a complete bipartite digraph } k\left(\frac{b}{2}, \frac{a}{2}\right) \text{ with vertex bipartition } (X_2, Y_2) \text{ such that } |X_2| = |Y_2| = \frac{b}{2}. \\
(D2) & \quad |(X_1, Y_2)_D \cup (Y_1, X_2)_D| = \left\lceil \frac{k}{2} \right\rceil \text{ and } |(X_2, Y_1)_D \cup (Y_2, X_1)_D| = \left\lfloor \frac{k}{2} \right\rfloor. \\
(D3) & \quad \text{for every vertex } u \in U, \text{ and for every } x' \in X_1 \text{ and } x'' \in X_2, \text{ we have both } (u, x'), (x'', u) \in A(D).
\end{align*} \]

From (D1),(D3), \( D \) is a bipartite digraph with vertex bipartition \((X,Y)\), where \( X = X_1 \cup X_2 \) and \( Y = Y_1 \cup Y_2 \cup U \). Moreover, \( D \) is non-supereulerian, since \(|(X_1, Y_2)_D \cup (Y_1, X_2)_D| = \frac{k}{2}\), and \(|\partial_D^+(A)| = \frac{k}{2}\). By (D3), \(|(U, B)_D| = 0\) and \(\tau(U) = \ell \geq \frac{k}{2} + 1\). By applying Proposition 2.10, it follows that

\[ h(U, A) = \left| \partial_D^+(A) \right| + \left| (U, B)_D \right| - \tau(U) = \frac{k}{2} - |U| < 0. \]

Thus \( h(U) < 0 \), and so by Proposition 2.10, \( D \) is not supereulerian.
The following is another necessary condition for a digraph to be supereulerian has been investigated by Alsatami et al. [7].

**Lemma 2.17** (K.A. Alsatami et al., Lemma 2 of [7]) A digraph $D$ is not supereulerian if for some integer $m > 0$, $V(D)$ has vertex disjoint subsets $\{B, B_1, \ldots, B_m\}$ satisfying both of the following:

i) $N_D(B_i) \subset B$, for all $i \in \{1, 2, \ldots, m\}$.

ii) $|\partial_D(B)| \leq m - 1$.

Lemma 2.17 has been helped many researchers to investigate the non-supereulerianicity for some families of digraphs, the following examples showed that.

**Example 2.18** [7] Let $n_1, n_2 \geq 3$ be integers and $C_{n_1} = v_{11}v_{12} \ldots v_{1n_1}v_{11}$ and $C_{n_2} = v_{21}v_{22} \ldots v_{2n_2}v_{12}$ be to dicycles of length $n_1$ and $n_2$, respectively, such that $V(C_{n_1}) \cap V(C_{n_2}) = \emptyset$. Consider $D'$ is a digraph obtained from $C_{n_1}$ and $C_{n_2}$ by identifying the arc $(v_{11}, v_{12})$ in $C_{n_1}$ with the arc $(v_{21}, v_{22})$ in $C_{n_2}$. Let $V(B) = \{v_{12}\}$, $V(B_1) = \{v_{13}\}$ and $V(B_2) = \{v_{23}\}$ be a subdigraphs of $D'$ . By applying Lemma 2.17, so $D'$ is non-supereulerian.
The families $\mathcal{F}(P_4, 5)$, $\mathcal{F}(P_4, 6)$ and $\mathcal{F}(P_4, 7)$ are defined at Definition 1.9. The following examples have been showed the families $\mathcal{F}(P_4, 5)$, $\mathcal{F}(P_4, 6)$ and $\mathcal{F}(P_4, 7)$ are non-supereulerian digraphs.

**Example 2.19** \([5]\) Let $M = xzy$ be a symmetric dipath, $Q = xuy$ be a dipath and $H_i = xv_iy$, $i \geq 1$ be dipaths. Let $D_1 = M \cup Q \cup H_1 \cup \{(u, z)\}$. For any $P_4$ in $D_1$, $|A(D[V(P_4)])| \geq 5$ and $|A(D[V(uvv_1)])| = 5$ and by Lemma 2.17 $B = \{x\}$, $B_1 = \{u\}$ and $B_2 = \{v_1\}$. Thus, $D_1$ is not supereulerian. Let $D_\ell = D_1 \cup \{H_2, \ldots, H_\ell\}$. Then $D_\ell \in \mathcal{F}(P_4, 5)$ and by Lemma 2.17, $D_\ell$ is non-supereulerian.

![Figure 8. The digraph family $D_\ell$](image)

**Example 2.20** \([5]\) Let $M = xzy$ be a symmetric dipath, $Q = xuy$ be a dipath and $H_i = xv_iy$, $i \geq 1$ be dipaths. Let $D_1 = M \cup Q \cup H_1$. For any $P_4$ in $D_1$, $|A(D[V(P_4)])| = 6$ and by Lemma 2.17 let $B = \{x\}$, $B_1 = \{u\}$ and $B_2 = \{v_1\}$. Thus, $D_1$ is not supereulerian. Let $D_\ell = D_1 \cup \{H_2, \ldots, H_\ell\}$. Then $D_\ell \in \mathcal{F}(P_4, 6)$ and by Lemma 2.17, $D_\ell$, is non-supereulerian.

![Figure 9. The digraph family $D_\ell$](image)

**Example 2.21** \([5]\) Let $M = xzy$ be a symmetric dipath, $Q = xuy$ be a dipath and $H_i = xv_iy$, $i \geq 1$ be dipaths. Let $D_1 = M \cup Q \cup \{\cup_{i=1}^{l} H_i\} \cup \{(x, y)\}$, $D_1 \in \mathcal{F}(P_4, 7)$. By Lemma 2.17, let $B = D[x]$, $B_1 = D[u]$ and $B_2 = D[v_1]$, we have $D_1$ is non-supereulerian.
As we mentioned on previous chapter for Definition 1.15 of product digraphs and Definition 1.12 of a cycle vertex cover of a digraph $D$, Alsatami et al. [6] used Lemma 2.17 to show that the Cartesian product of supereulerian digraph $D_1$ and a strong digraph $D_2$, which has an eulerian vertex cover with $m$ eulerian subdigraphs and $m > |V(D_1)|$, that the Cartesian product $D_1 \square D_2$ is non-supereulerian.

**Example 2.22** [6] Let $D_1$ be a supereulerian digraph with $V(D_1) = \{u_1, u_2\}$ and $A(D_1) = \{(u_1, u_2), (u_2, u_1)\}$. Let $D_2$ be a strong digraph with $V(D_2) = \{v_1, v_2, v_3, v_4, v_5\}$ and $A(D_2) = \{(v_2, v_1), (v_1, v_3), (v_3, v_2), (v_1, v_4), (v_4, v_2), (v_1, v_5), (v_5, v_2)\}$, which has an eulerian vertex cover with 3 eulerian subdigraphs. By Definition 1.15, we can obtain the Cartesian product $D_1 \square D_2$ of $D_1$ and $D_2$ (See Fig. 10). Let $B, B_1, B_2$ and $B_3$ be vertex-disjoint subsets of $V(D_1 \times D_2)$ with $B = \{(u_1, v_1), (u_2, v_1)\}$, $B_1 = \{(u_1, v_3), (u_2, v_3)\}$, $B_2 = \{(u_1, v_4), (u_2, v_4)\}$ and $B_3 = \{(u_1, v_5), (u_2, v_5)\}$. We find that $N_D^{-1}(B_i) \subset B$ for $i \in \{1, 2, 3\}$ and $|\partial_D^{-1}(B)| = 2$. By Lemma 2.17, the Cartesian product $D_1 \square D_2$ is non-supereulerian.

![Diagram](image)

Figure 10. $D_1 \square D_2$

The following two examples have been used Lemma 2.17 to show that the extended digraph of an eulerian digraph and the digraphs under some degree condition are non-supereuleian.

**Example 2.23** [23] (Extended digraphs) Let $D$ be an eulerian digraph with $V(D) = \{v_1, v_2, ..., v_8\}$ and let $D'$ be a digraph obtained from of $D$ by splattering one vertex say $v_5$ to $v'_5$ and $v''_5$ such that $N_{D'}^{-1}(v'_5) = N_D^{-1}(v'_5) = N_D(v'_5)$ and $N_D^{-1}(v''_5) = N_D(v''_5) = N_D(v)$, so $V(D') = \{v_1, v_2, v_3, v_4, v'_5, v''_5, v_6, v_7, v_8\}$ (see Fig. 13). Let $B, B_1, B_2, B_3$ be vertex disjoint subsets of $V(D')$ with $B = \{v_4\}$, $B_1 = \{v_1, v_2, v_3\}$, $B_2 = \{v'_5\}$ and $B_3 = \{v''_5\}$. We find that $N_D^{-1}(B_i) \subset B$ for $i \in \{1, 2, 3\}$ and $|\partial_D^{-1}(B)| = 2$. By Lemma 2.17, the digraph $D'$ is non-supereulerian.
An eulerian digraph $D$

Figure 11. The digraph $D$ and $D'$

Example 2.24 [1] Let $G, H$ be two digraphs isomorphic to $K^*_m$, where $m \geq 2$. Let $u, x \in V(G)$ and $v, y \in V(H)$. Let $D_m = G \cup H \cup \{(z_1, u), (z_1, v), (x, z_2), (y, z_2), (z_2, z_1)\}$. Then $V(D_m) = n = 2m + 2$. (See Fig. 12. for $m = 3$). By Lemma 2.17 with $A = D[z_1], B_1 = G$ and $B_2 = H$, we conclude that $D_m$ is not supereulerian eventhough $d_D^+(x) + d_D^+(y) + d_D^+(u) + d_D^+(v) = 4m = 2n - 4$.

Finally, there is another necessary condition of some specific digraphs to be supereulerian which is also sufficient condition. Follows from the definition of semicompete multipartite digraphs and Definition 1.13 of a cycle factor, Bang-Jensen and Maddaloni [10] proved the following theorem for a semicomplete multipartite digraph to be supereulerian.

Theorem 2.25 [10] Let $D$ be a semicomplete multipartite digraph. Then $D$ is supereulerian if and only if it is strong and has a cycle factor.

Next example showed the existences of a cycle factor is the necessary condition of a strong semicomplete multipartite digraphs to be supereulerian.

Example 2.26 [10] Let $D$ be the semicomplete multipartite digraph with five partitesets $U, W, W', Z, Z'$, where $U$ has size $k + 1$ and the others have size $k$. $W$ has all the possible arcs from all the other partite sets and so does $W'$. $Z$ has all the possible arcs to all the other partite sets and so does $Z'$. Moreover there
is a matching from $W$ to $Z$. Since $D$ has no cycle factor; then by Theorem 2.25, $D$ is not supereulerian. (Fig. 13. shows an example with $k = 3$ where the thick arcs between sets represent complete adjacency in the direction of the arc, double arcs indicate arcs in both directions).

![Diagram of a non-supereulerian semicomplete multipartite digraph](image)

Figure 13. A non-supereulerian semicomplete multipartite digraph $D$ with $\alpha(D) = 3$ and $\lambda(D) = 2$.

Follows Definition 1.6 of a locally semicomplete multipartite digraphs, F. Liu, Z-X. Tian, D. Li [38] generalized the result of Bang-Jensen and Maddaloni for a semicomplete multiparticle digraph that they used the same approach that Bang-Jensen and Maddaloni used in [10] and they proved the following result.

**Theorem 2.27** (Liu, Tian and Li, Theorem 2.5 of [38]) Let $D$ be a locally semicomplete multipartite digraph. Then $D$ is supereulerian if and only if it is strong and has a cycle factor.

Follows from Definition 1.7 of a quasi-transitive digraph and Definition 1.13 of a cycle factor, the following theorem has been proved by [10] of any quasi-transitive digraphs to be supereulerian. In [10] proved that the existences of a cycle factor if the necessary condition of a strong quasi-transitive digraphs to be supereulerian and it is a sufficient condition as well.

**Theorem 2.28** (Bang-Jenson and Maddaloni, Theorem 2.12 of [10]) Let $D$ be a quasi-transitive digraph. $D$ is supereulerian if and only if it is strong, with canonical decomposition $D = S[Q_1, \ldots, Q_s]$, and the semicomplete directed multigraph $S_1$ obtained from $D$ by contracting each $Q_i$ into a single vertex $v_i$ has an cycle factor $E'$ such that $d^+_D[E'](v_i) \geq \tau(Q_i)$ for every $i = 1, \ldots, s$. 18
Next example showed a existences of a cycle factor is a necessary condition of a strong quasi-transitive digraphs to be supereulerian.

**Example 2.29** [10] Let $D$ be the quasi-transitive digraph with vertex set given by an independent set $U$ on $k$ vertices, together with two complete digraphs $W, Z$ on $k - 1$ vertices and all the arcs from $U$ to $W$, all the arcs from $Z$ to $W \cup U$ and a matching from $W$ to $Z$. Since $D$ does not even have a cycle factor; then by Theorem 2.28, $D$ is not supereulerian. (Fig. 14. shows an example with $k = 3$ where the thick arcs between sets represent complete adjacency in the direction of the arc, double arcs indicate arcs in both directions).

![Figure 14. A non-supereulerian semicomplete multipartite digraph $D$ with $\alpha(D) = 3$ and $\lambda(D) = 2$.](image)

**Theorem 2.30** (Dong and Liu, Thorem1.3 of [23]) An extended cycle $D'$ is supereulerian if and only if $D'$ is strong and has a cycle factor.

C. Dong et al. in [24] gave a necessary and a sufficient conditions involving 3-path-quasi-transitive digraphs to be supereulerian.

The 3-path-quasi-transitive digraphs are defined in Definition 1.11 where the following theorem is a necessary condition of the 3-path-quasi-transitive digraphs to be supereulerian and it is also a sufficient condition. In [24] proved the following theorem to for each $i \in \{1, 2, 3, 4\}$.

**Theorem 2.31** [24] Let $D$ be a strong $\mathcal{H}_i$-quasi-transitive digraph, then $D$ is supereulerian if and only if $D$ contains a cycle factor.

### 2.3 Degree Condition for Supereulerian Digraphs

In this section will give the brief discussion of sufficient degree conditions for supereulerian of digraphs. One of the motivation of the studies of supereulerian digraphs is the study of hamiltonian digraphs, as
hamiltonian graphs are also supereulerian. We start with the main origin of the degree condition idea, Dirac condition and Ore conditions, which are commonly used to study hamiltonian (di)graphs. For any graph $G$, a path that contains every vertex of $G$ is called a Hamilton path of $G$; similarly, a Hamilton cycle of $G$ is a cycle that contains every vertex of $G$. A graph is hamiltonian if it contains a Hamilton cycle. The result of Dirac in 1952 introduced in [15] as sufficient conditions for a graph $G$ to be hamiltonian, which is a useful result of hamiltonian graphs.

**Theorem 2.32** (Dirac’s Theorem)[15] If $G$ is a simple graph with $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then $G$ is hamiltonian.

Ore [41] generalized the previous theorem to introduce the degree condition of graphs to be hamiltonian.

**Theorem 2.33** (Ore’s Theorem)[41] A graph satisfying $d(x) + d(y) \geq n$ for every pair $x, y$ of nonadjacent vertices is hamiltonian.

As it is the case for undirected graphs, some sufficient degree conditions for hamiltonicity in digraphs can be (slightly) weakened to become sharp sufficient conditions for supereulerianity. The property of being supereulerian is at the same time relaxation of being hamiltonian: being supereulerian digraph means having a closed ditrail covering all the vertices of the digraph; being hamiltonian means having a closed ditrail covering all vertices of the digraph without using a vertex twice. In this section, we display some sufficient conditions for a digraph to be supereulerian. For a digraph part, there are many results of digraphs to be hamiltonian.

**Theorem 2.34** (Nash-Williams)[40] Let $D$ be a digraph of order $n \geq 3$ such that for every vertex $x$, $d^{+}(x) \geq \frac{n}{2}$ and $d^{-}(x) \geq \frac{n}{2}$, then $D$ is hamiltonian.

**Theorem 2.35** (Ghouila-Houri)[27] Let $D$ be a strongly connected digraph of order $n \geq 3$. If $d(x) \geq n$ for all vertices $x \in V(D)$, then $D$ is hamiltonian.

**Theorem 2.36** (Woodall)[45] Let $D$ be a digraph of order $n \geq 3$. If $d^{+}(x) + d^{-}(y) \geq n$ for all pair of non-adjacent vertices, then $D$ is hamiltonian.

There are two generalization of Woodall theorem. The first generalization by Meyniel.

**Theorem 2.37** (Meyniel)[39] Let $D$ be a strongly connected digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices in $D$, then $D$ is hamiltonian.

The second generalization by Bong-Jenson, Gutin and Li in [12].

**Theorem 2.38** (Bang-Jensen, Gutin, Li, Theorem 4.1 of [12]) Let $D$ be a strongly connected digraph of order $n \geq 2$. Suppose that $\min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for every pair of non-adjacent vertices $x, y$ with a common in-neighbor. Then $D$ is hamiltonian.
Bang-Jensen, Maddaloni[10] proved the analogue of Meyniel’s theorem for supereulerian part which is the degree condition for digraphs to be supereulerian, where they gave some sufficient Ore-type conditions to be supereulerian. In the theorems below, we always assume $D$ is a digraph on $n$ vertices. A pair of vertices $x$ and $y$ are adjacent in $D$ if $(x,y)$ or $(y,x)$ is in $A(D)$.

**Theorem 2.39** (Bang-Jensen, Maddaloni, Theorem 3.6 of [10]) A strong digraph such that $d(x) + d(y) \geq 2n - 3$ for all of non-adjacent vertices $x, y$ is supereulerian.

In [30], Y. Hong, H. Lai, Q. Liu define the the family $D_0(k_1, k_2, 2)$ is the set of spanning subdigraphs $D'$ of the digraphs $D$ in $\mathcal{D}(k_1, k_2, 2)$ defined in Example 2.11, which satisfy $\delta^+(D') + \delta^-(D') = |V(D')| - 4$. Y. Hong et al. [30] proved that no digraph in $D \in D_0(k_1, k_2, 2)$ has a spanning eulerian subdigraph. Moreover, Y. Hong, H. Lai, Q. Liu [30] investigated the Ore-type sufficient condition of supereulerian digraphs and proved the following theorem.

**Theorem 2.40** (Hong, Lai, Liu, Theorem 3.4 of [30]) Let $D$ be a strong digraph of order $n$ and minimum out-degree $\delta^+(D) \geq 4$ and minimum in-degree $\delta^-(D) \geq 4$. If $\delta^+(D) + \delta^-(D) \geq n - 4$, then the following are equivalent.

(i) $D$ has a spanning eulerian subdigraph.

(ii) Either $\delta^+(D) + \delta^-(D) > n - 4$, or for some integer $k_1, k_2, \delta^+(D) = k_1, \delta^-(D) = k_2$ but $D \notin D_0(k_1, k_2, 2)$.

Follows from the previous theorem, Hong et al in [30] showed that Example 2.11 shows that the bound in Theorem 2.40 is a best possible lower bound of the minimum degree.

There are other degree conditions for supereulerian digraphs. Another Ore-type condition has been investigated. Y. Hong, H. Lai, Q. Liu [31] characterized families of digraphs, let $\mathcal{D}_1$ be the family $\mathcal{D}(k_1, k_2, 2)$ as defined in Example 2.11 which proved that a simple digraph $D$ satisfying $\min\{\delta^+(D), \delta^-(D)\} \geq 4$ and $\delta^+(D) + \delta^-(D) \geq n - 4$, then $D$ is supereulerian if and only if $D$ is not a member in $\mathcal{D}_1$.

Let $\mathcal{D}_2$ as defined in Example 2.12 which is non-supereulerian and let $\mathcal{D}_3 \subset \mathcal{D}(0, k_2, 2) \cup \mathcal{D}(k_1, 0, 2)$ as Example 2.13. Thus for $i = 1, 2, 3$ none of the spanning subdigraphs of digraphs in $\mathcal{D}_i$ has a spanning eulerian subdigraph. Y. Hong in [31] defined that for $i = 1, 2, 3$, let $\mathcal{F}_i$ be the family of digraphs such that $D \in \mathcal{F}_i$ if and only if for some member $D' \in \mathcal{D}_i$, $D$ is a strong spanning subdigraph of $D'$ satisfying $d'_D(x) + d'_D(y) \geq n - 4$ for any pair of vertices $x, y$ with $xy \notin A(D)$. Then, each $\mathcal{F}_i$ is also a family of non-supereulerian digraphs, it follows the following theorem.

**Theorem 2.41** (Hong, Liu, Lai, Theorem 3.4 of [31]) Let $D$ be a strong digraph of order $n \geq 11$. If $d'_D(u) + d'_D(v) \geq n - 4$ for any pair of vertices $u, v$ with $(u, v) \notin A(D)$, then $D$ is supereulerian if and only if it $D \notin \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Recall that an ordered pair of vertices $x, y$ is dominated (dominating, respectively) if there exists $z \in V(D)$, with $(z, x), (z, y) \in A(D)((x, z), (y, z) \in A(D)$, respectively). Next theorem is due to Zhao and Meng.

**Theorem 2.42** [49] Let $D$ be a strong digraph of order $n \geq 2$. If $d'_D(x) + d'_D(y) + d'_D(u) + d'_D(v) \geq 2n - 1$ for every pair $x, y$ of dominating vertices and every pair $u, v$ of dominated vertices, then $D$ is hamiltonian.
Algefari [1] studied this kind of sufficient conditions in Theorem 2.42, for a digraph to be supereulerian, and proved the following theorem.

**Theorem 2.43** [1] Let $D$ be a strong digraph of order $n \geq 2$. If $d_D^+(x) + d_D^+(y) + d_D^-(u) + d_D^-(v) \geq 2n - 3$ for every pair $x, y$ of dominating non-adjacent vertices and every pair $u, v$ of dominated non-adjacent vertices, then $D$ is supereulerian.

In addition, Algefari [1] define infinite family of nonsupereulerian digraphs as seen in Example 2.24 which makes Theorem 2.43 sharp.

### 2.4 Bang-Jensen and Thomassé Conjecture for Digraphs to be Supereulerian

In this section, we start with a well known theorem of Chvátal Erdös [21] states that every 2-connected graph $G$ with $\kappa(G) \geq \alpha(G)$ is hamiltonian. Thomassen [44] gave an infinite family of non-hamiltonian (but supereulerian) digraphs such that $\kappa(D) = \alpha(D) = 2$, showing that the the Chvátal Erdös theorem does not extend to digraphs. This result motivates Bang-Jensen and Thomassé (2011, unpublished, see [11]) to make the following conjecture.

**Conjecture 2.44** Let $D$ be a digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.

Bang-Jensen and Maddaloni [10] indicated that the above condition is not necessary, and considered a directed cycle on four vertices $C_4$ as an example that where $C_4$ is eulerian digraph, and hence supereulerian, but $\lambda(C_4) = 1$ and $\alpha(C_4) = 2$. Moreover, they showed that Conjecture 2.44 is true for undirected graph.

**Theorem 2.45** (Bang-Jensen, Maddaloni, Theorem 2.3 of [10]) Let $G$ be an undirected graph on at least three vertices. If $\lambda(D) \geq \alpha(D)$, then $G$ is supereulerian.

Conjecture 2.44 has motivated many researchers to verified it for many digraph families. Let start with Bang-Jensen and Maddaloni [10], who proved that Conjecture 2.44 is true for semicomplete multipartite digraphs and for quasi-transitive digraphs.

**Theorem 2.46** (Bang-Jensen, Maddaloni, Theorem 2.10 of [10]) Let $D$ be a semicomplete multipartite digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.

**Theorem 2.47** (Bang-Jensen, Maddaloni, Theorem 2.13 of [10]) Let $D$ be a quasi-transitive digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.

Bang-Jensen and Maddaloni [10] proved the following useful theorem where they used flow theory to show that the condition $\lambda(D) \geq \alpha(D)$ guarantees the existence of a cycle factor. The follow is used to prove Theorem 2.25 and Theorem 2.28.
Theorem 2.48 (Bang-Jensen, Maddaloni, Theorem 2.4 of [10]) Let \( D \) be a digraph. If \( \lambda(D) \geq \alpha(D) \), then \( D \) has a cycle factor.

Bang-Jensen and Maddaloni [10] provided Example 2.26 and Example 2.29 to show that there exists infinite families of digraphs with \( \lambda(D) \geq \alpha(D) - 1 \) that are not supereulerian. Hence, Example 2.26, Example 2.29, respectively, showed that Conjecture 2.44 would be best possible for both semicomplete multipartite digraphs and quasi-transitive digraphs.

Following definition of locally semicomplete multipartite digraph, Definition 1.6, F. Liu, Z. Xian, D. Li [38] generalized the result of Bang-Jensen and Maddaloni for a semicomplete multpartite digraph and they proved the following result for a locally semicomplete multipartite digraphs, they used the same approach that Bang-Jensen and Maddaloni used in [10] where F. Liu, et al.[38] used Theorem 2.48 and Theorem 2.27 to drive the following theorem.

Theorem 2.49 (Liu, Xian, Li, Theorem 2.6 of [38]) Let \( D \) be a locally semicomplete multipartite digraph. If \( \lambda(D) \geq \alpha(D) \), then \( D \) is supereulerian.

Following the definition of 3-path-quasi-transitive digraphs provided in Definition 1.11, Dong, Liu, Meng,[24], showed that Conjecture 2.44 has been verified for 3-path-quasi-transitive in [24], where the following theorem to for each \( i \in \{1, 2, 3, 4\} \).

Theorem 2.50 (Dong, Liu, Meng, Theorem 1.2 of [24]) Let \( D \) be a strong \( H_i \)-quasi-transitive digraph for \( i \in \{1, 2, 3, 4\} \). If \( \lambda(D) \geq \alpha(D) \), then \( D \) is supereulerian.

C. Dong et al. [24] have used Theorem 2.48 and Theorem 2.31 to prove that Conjecture 2.44 is true for 3-path-quasi-transitive digraphs.

Many other researchers have investigated Conjecture 2.44. In particular, Algefari et al.[2] proved the following result.

Theorem 2.51 (Algefari, Lai, Xu, Theorem 1.5 of [2]) Let \( D \) be a strong digraph. If \( \lambda(D) \geq \alpha'(D) \), then \( D \) is supereulerian.

As \( \alpha'(D) = \alpha'(G(D)) \), Algefari et al. [2] used the following fundamental theorem of graph theory to prove Theorem 2.51.

Theorem 2.52 (Berge, 1957)[14] A matching \( M \) in \( G \) is a maximum matching if and only if \( G \) does not have \( M \)– augmenting paths.

X. D. Zhang, J. Liu, L. Wang, H.-J. Lai [48] proved that Conjecture 2.44 holds for a bipartite digraph with the lower bound begin half of the conjecture bound by proving the following result.

Theorem 2.53 (Zhang, Liu, Lai, Theorem 1.5 of [48]) Let \( D \) be a strong bipartite digraph. If \( \lambda(D) \geq \left\lceil \frac{\alpha(D)}{2} \right\rceil + 1 \), then \( D \) is supereulerian.
X. Zhang et al. [48] provided the following theorem as a tool to prove Theorem 2.53.

**Theorem 2.54** (Zhang, Liu, Lai, Theorem 1.4 of [48]) Let $D$ be a strong bipartite digraph with a vertex bipartition $(X,Y)$ satisfying $|X| \leq |Y|$. Each of the following holds.

(i) If $\delta(D) \geq \left\lceil \frac{\alpha'(D)}{2} \right\rceil + 1$, then $D$ is supereulerian.

(ii) Suppose that $\alpha'(D)$ is even and $\alpha'(D) < |X|$. If $\delta(D) \geq \frac{\alpha'(D)}{2}$, then $D$ is supereulerian.

As $\alpha(D) \geq |Y| \geq |X| \geq \alpha'(D)$, X. Zhang et al. [48] concluded that Theorem 2.53 follows from Theorem 2.54 (i). Also, as $\delta(D) \geq \lambda(D) \geq \kappa(D)$, thus $\delta(D)$ can be replaced by either $\lambda(D)$ or $\kappa(D)$ in Theorem 2.54. Moreover, Example 2.16 showed that Theorem 2.53 is sharp in some sense of non-supereulerian strong bipartite digraphs.

In [23], Bang-Jensen and Thomasse’s conjecture has also been verified for several extended digraph such as extended hamiltonian, an arc-locally semicomplete digraph, an extended arc-locally semicomplete digraph.

### 2.5 Supereulerian Digraphs with Global or Local Density Conditions

In this section, we will introduce some local structures of some digraphs to be supereulerian. The following theorem proved by [4].

**Theorem 2.55** (Algefari, Alsatami, Lai, Liu, Theorem 1.3 (i) of [4]) Every symmetrically connected digraph is supereulerian.

Follows from Definition 1.9, Algefari et al. in [5] observed that if $D \in \mathcal{F}(P_2, 2) \cup \mathcal{F}(P_3, 5)$, then $D$ is symmetrically connected, and so by Theorem 2.55, every digraph in $\mathcal{F}(P_2, 2) \cup \mathcal{F}(P_3, 5)$ is supereulerian.

**Theorem 2.56** (Algefari, Alsatami, Lai, Liu, Theorem 1.3 (ii) of [4]) Every partially symmetric digraph is supereulerian.

Another result has been proved by Algefari, Lai, Liu and Zhang [5] who studied the supereulerianicity of digraphs in $\mathcal{F}(P_4, h)$, and determined the smallest value of $h_4$ such that every digraph in $\mathcal{F}(P_4, h_4)$ is supereulerian by proving the following theorem.

**Theorem 2.57** (Algefari et al, Theorem 3.1 (i) of [5]) Every digraph $D$ in $\mathcal{F}(P_4, 8)$ is supereulerian.

As in Example 2.21 showed that there exist at least one non-supereulerian digraph in $\mathcal{F}(P_4, 7)$ which showed that Theorem 2.57 is sharp in some sense.

As well known, for any digraph $D$, $0 \leq \text{diam}(D) \leq \infty$. If a digraph $D$ with $\text{diam}(D) = 0$, that is, $D \cong k^*_1$, then $D$ is supereulerian. If a digraph $D$ on $n > 1$ vertices with $\text{diam}(D) = 1$, that is, $D \cong k^*_n$, then $D$ is supereulerian. In 2018, C. Dong, J. Liu, X. Zhang [22] obtained sufficient condition on digraphs to be supereulerian for a given diameter.
Theorem 2.58 (Dong, Liu and Zhang, Theorem 3.1 of [22]) A digraph \( D \) with \(|V(D)| \geq 3 \) and \( \text{diam}(D) \leq 2 \) is supereulerian.

Moreover, Example 2.14 indicated that there are infinitely many non-supereulerian digraphs with \( \text{diam}(D) = 3 \), so Theorem 2.58 is sharp in some sense.

Another result provided in [22], they discussed the supereulerian bipartite digraph with diameter 3 and proved the following theorem of bipartite digraph.

Theorem 2.59 (Dong, Liu and Zhang, Theorem 4.1 of [22]) A bipartite digraph \( D \) with \(|V(D)| \geq 4 \) and \( \text{diam}(D) \leq 3 \) is supereulerian.

2.6 Supereulerian Sums and Products of Digraphs

In this section, we introduce the definition of 2-sum digraph and display results of sufficient conditions of 2-sum digraph and product of two digraphs \( D_1, D_2 \) to be supereulerian.

2.6.1 Digraph 2-Sum


Definition 2.60 Let \( D_1 \) and \( D_2 \) be two vertex disjoint digraphs, and let \( a_1 = (v_{11},v_{12}) \in A(D_1) \) and \( a_2 = (v_{21},v_{22}) \in A(D_2) \) be two distinguished arcs. The 2-sum \( D_1 \oplus a_1, a_2 D_2 \) of \( D_1 \) and \( D_2 \) with base arcs \( a_1 \) and \( a_2 \) is obtained from the union of \( D_1 \) and \( D_2 - a_2 \) by identifying \( v_{11} \) with \( v_{21} \) and \( v_{12} \) with \( v_{22} \), respectively. When the arcs \( a_1 \) and \( a_2 \) are not emphasized or is understood from the context, often used \( D_1 \oplus_2 D_2 \) for \( D_1 \oplus a_1, a_2 D_2 \).

By Definition 2.60, \( D' \) in Example 2.18 is \( C_{n_1} \oplus_2 C_{n_2} = C_{n_1} \oplus_{a_1,a_2} C_{n_2} \) such that \( a_1 = (v_{11},v_{12}) \) and \( a_2 = (v_{21},v_{22}) \) which is non-supereulerian. Alsatami et al. in [7] obtained several sufficient conditions on \( D_1 \) and \( D_2 \) for \( D_1 \oplus a_1, a_2 D_2 \) to be supereulerian. In particular, they showed that if \( D_1 \) and \( D_2 \) are symmetrically connected or partially symmetric, then \( D_1 \oplus a_1, a_2 D_2 \) is supereulerian. Their main result of this direction, is to show that the digraph 2-sums of symmetrically connected or partially symmetric digraphs are supereulerian. The following lemma has been proved in [7].

Lemma 2.61 [7] Let \( D_1 \) and \( D_2 \) be two vertex disjoint digraphs with \( a_1 = (v_{11},v_{12}) \in A(D_1) \) and \( a_2 = (v_{21},v_{22}) \in A(D_2) \) and let \( C_{n_1} \oplus_2 C_{n_2} \) denote \( D_1 \oplus a_1, a_2 D_2 \). Each of the following holds.

i) If \( D_1 \) and \( D_2 \) are symmetrically connected, then \( D_1 \oplus_{a_1,a_2} D_2 \) is symmetrically connected.

ii) If \( D_1 \) and \( D_2 \) are partially symmetric, then \( D_1 \oplus_{a_1,a_2} D_2 \) is partially symmetric.

iii) If \( D_1 \) is symmetric and \( D_2 \) is partially symmetric, then \( D_1 \oplus_{a_1,a_2} D_2 \) is partially symmetric.

By using Theorem 2.55 and Theorem 2.56 with Lemma 2.61, then the following has been proved.
Theorem 2.62  (K. A. alsatami et al., Theorem 4 of [7]) Let $D_1$ and $D_2$ be two digraphs. Each of the following holds.
(i) If $D_1$ and $D_2$ are symmetrically connected, then $D_1 \oplus D_2$ is supereulerian.
(ii) If $D_1$ and $D_2$ are partially symmetric, then $D_1 \oplus D_2$ is supereulerian.
(iii) If $D_1$ is symmetric and $D_2$ is partially symmetric, then $D_1 \oplus D_2$ is supereulerian.

2.6.2 Product Digraph

In [26], an open problem (Problem 6 of [26]) was raised to find natural conditions for the product of graphs to be hamiltonian. Motivated by this problem, K.A. Alsatami, J. Liu and X.D. Zhang [6], proposed to seek natural conditions on digraphs $D_1$ and $D_2$ such that the product of $D_1$ and $D_2$ is supereulerian. K.A. Alsatami et al. [6] investigated sufficient conditions on $D_1$ and $D_2$ for $D_1 \square D_2$ and $D_1[D_2]$ to be supereulerian or trailable investigated. The following useful theorem has been used as a tool to show the results of K. Alsatami et al.[6].

Theorem 2.63  [47] Let $D_1$ and $D_2$ be eulerian digraphs. Then the Cartesian product $D_1 \square D_2$ is eulerian.

K. Alsatami et al.[6] have been proved the following theorem, whose sharpness is showed in Example 2.22.

Theorem 2.64  (Alsatami, Liu and Zhang, Theorem 2.3 of [6]) Let $D_1$ and $D_2$ be two strong digraphs with $\min\{|V(D_1)|,|V(D_2)|\} \geq 2$ such that $D_1$ is supereulerian and $D_2$ has an eulerian vertex cover with $m$ eulerian subdigraphs such that $m \leq |V(D_1)|$. Then the Cartesian product $D_1 \square D_2$ is supereulerian.

Corollary 2.65  [6] Let $D_1$ be a supereulerian digraph and $D_2$ be a digraph.
(i) If $D_2$ is supereulerian, then the Cartesian product $D_1 \square D_2$ is supereulerian.
(ii) If $D_2$ is trailable, then the Cartesian product $D_1 \square D_2$ is trailable.

Follows from Definition 1.15(v) of the Lexicographic product $D_1[D_2]$ of two digraphs $D_1$ and $D_2$, the following two results have proved by [6].

Theorem 2.66  (Alsatami, Liu and Zhang, Theorem 2.5 of [6]) Let $D_1$ and $D_2$ be two digraphs. If $D_1$ is supereulerian with $|V(D_1)| \geq 2$, then the Lexicographic product $D_1[D_2]$ is supereulerian.

Theorem 2.67  (Alsatami, Liu and Zhang, Theorem 2.6 of [6]) Let $D_1$ and $D_2$ be two strong digraphs with $\min\{|V(D_1)|,|V(D_2)|\} \geq 2$ such that $D_1$ is trailable. Then the Lexicographic product $D_1[D_2]$ is supereulerian.

Follows from Definition 1.15(iii) of the Strong product digraph $D_1 \boxtimes D_2$ of digraphs $D_1$ and $D_2$, the following results has been verified in this dissertation.
Theorem 2.68 (H-J Lai et al., Theorem 1.6 of [36]) Let $D_1$ and $D_2$ be strong digraphs. If for some cycle factor $F$ of $D_1$, $D_1/F$ is hamiltonian with $f(D_2) \leq |V(D_1)|$, then the strong product $D_1 \boxtimes D_2$ is supereulerian.
Chapter 3

3 Matching and Spanning Trail in Digraphs

In this chapter, we motivated the result of Bang-Jensen and Thomassé conjecture 2.44; if \( \lambda(D) \geq \alpha(D) \), then \( D \) is supereulerian. Algefari et al in [2], motivated Bang-Jensen and Thomassé conjecture and proved Theorem 2.51 in the previous chapter, for a strong digraph \( D \); if \( \lambda(D) \geq \alpha'(D) \), then \( D \) is supereulerian. This motivates us to study for strong digraphs with \( \lambda(D) \geq \alpha'(D) - 1 \) and we show the following theorem which is the main result of this chapter.

Theorem 3.1 Let \( D \) be a strong digraph on \( n \geq 12 \) vertices satisfying \( \lambda(D) \geq \alpha'(D) - 1 \). Each of the following holds.
(i) There exists a family \( D \) of well-characterized digraphs such that for any digraph \( D \) with \( \alpha'(D) \leq 2 \), \( D \) has a spanning trail if and only if \( D \) is not a member in \( D \).
(ii) If \( \alpha'(D) \geq 3 \), then \( D \) has a spanning trail.
(iii) If \( \alpha'(D) \geq 3 \) and \( n \geq 2\alpha'(D) + 3 \), then \( D \) is supereulerian.
(iv) If \( \lambda(D) \geq \alpha'(D) \geq 4 \) and \( n \geq 2\alpha'(D) + 3 \), then for any pair of vertices \( u \) and \( v \) of \( D \), \( D \) contains a spanning \((u,v)\)-trail.

3.1 The symmetric core of digraphs

In this section, we introduce the symmetric core of digraphs and some of its proprieties. We use \( \mathbb{Z}_n \) to denote the (additive) group of integers modulo \( n \).

Definition 3.2 [37] For a digraph \( D \), an arc \([u,v] \in A(D)\) is a symmetric in \( D \) if both arcs \((u,v)\) and \((v,u)\) are in \( A(D) \). Let \( S(D) = \{e \in A(D) : e \text{ is symmetric in } D\} \). A digraph \( D \) is a symmetric if \( A(D) = S(D) \). The symmetric core of \( D \), denoted by \( J(D) \), has vertex set \( V(D) \) and arc set \( S(D) \).

Lemma 3.3 Let \( D \) be a digraph, \( J = J(D) \) and \( J_0 \) be a symmetric subdigraph of \( J \).
(i) For any \( v \in V(J_0) \), \( d^+_J(v) = d^+_J(v) \).
(ii) If \( J_0 \) is connected, then \( J_0 \) is an eulerian subdigraph of \( D \) and so \( J_0 \) is strongly connected.
(iii) Suppose that \( J_0 \) is connected. Then for any vertices \( u,v \in V(J_0) \), \( J_0 \) contains a spanning \((u,v)\)-trail.
(iv) If \( D \) is strong and for some vertices \( u,v \in V(D) \), \( D \) has a \((u,v)\)-trail \( P \) such that \( D - A(P) \) contains a connected symmetric subdigraph \( J' \) of \( J \) such that \( V(P) \cup V(J') = V(D) \), \( u,v \notin V(J') \) and there exist two vertices \( v^+, v^- \in V(J') \) with \( (v,v^+), (v^-, v) \in A(D) \), then \( D \) is supereulerian.
(v) If \( D/J_0 \) has a hamiltonian cycle, then \( D \) is supereulerian. In particular, if \( D \) is strong and \( J_0 \) is a spanning subdigraf of \( D \) with at most two connected components, then \( D \) is supereulerian.
(vi) If \( D \) is strong and \( D[A(D) - A(J_0)] \) has a trail \( T' \) that intersects every component of \( J_0 \) with \( V(D) - V(J_0) \subseteq V(T') \), then \( T = D[A(T') \cup A(J_0)] \) is a spanning trail in \( D \).
(vii) Suppose \( \lambda(D) \geq 2 \). If \( G(D - V(J_0)) \) is spanned by a 3-cycle, then \( D \) is supereulerian.
Proof. As (i) and (ii) are immediate consequences of the definitions, it suffices to justify the other conclusions. Let \( u, v \in V(J_0) \). By (ii), we assume that \( J_0 \) is strong and \( u \neq v \). Let \( P \) be a shortest \((v, u)\)-path in \( J_0 \). As \( P \) is shortest, if an arc \( e = (x, y) \in A(P) \), then \((y, x) \notin A(P)\). By (i), \( T = J_0 - A(P) \) is a connected digraph such that \( d_T^+(u) = d_T^-(u) + 1, d_T^+(v) = d_T^-(v) - 1 \) and for any vertex \( w \in V(T) - \{u, v\} \), \( d_T^+(w) = d_T^-(w) \). Thus \( T \) is a spanning \((u, v)\)-trail of \( J_0 \). This proves (iii).

By assumption, \( J' \) is a connected symmetric subdigraph, and so \( J' \) is the symmetric core of itself. By (iii) with \( J_0 = J' \), \( J' \) contains a spanning \((v^+, v^-)\)-trail \( T \). As \( A(T) \cap A(P) \subseteq A(J') \cap A(P) = \emptyset \), the arc set \( A(T) \cup A(P) \cup \{(v, v^+), (v^-, u)\} \) induces a spanning closed trail of \( D \), and so \( D \) is supereularian. Hence (iv) is justified.

To prove (v), let \( D' = D/J_0 \) and denote \( n = |V(D')| \). Suppose that \( D' \) has a Hamilton cycle \( C \) with \( V(C) = \{v_1, v_2, \ldots, v_n\} \) and \( A(C) = \{e_i = (v_i, v_{i+1}) : i \in \mathbb{Z}_n\} \). Let \( J_1, J_2, \ldots, J_n \) be the preimage of \( v_1, v_2, \ldots, v_n \), respectively. By definition, each \( J_i \) is a connected component of \( J_0 \), and so a connected symmetric subdigraph of \( J \). By the definition of contraction, \( A(D') \subseteq A(D) \), and so for each \( i \in \mathbb{Z}_n \), the arc \( e_i \in A(D) \). Therefore, there exist vertices \( v'_i \in V(J_i) \) and \( v''_{i+1} \in V(J_{i+1}) \) with \( e_i = (v'_i, v''_{i+1}) \in A(D) \). Since each \( J_i \) is a connected symmetric subdigraph of \( J \), it follows by (iii) that \( J_i \) has a spanning \((v'_i, v''_i)\)-trail \( T_i \). Let \( A_1 = \{(v'_i, v''_i) : i \in \mathbb{Z}_n\} \). Then \( H = D[A_1 \cup \bigcup_{i \in \mathbb{Z}_n} A(T_i)] \) is a spanning closed trail of \( D \), and so \( D \) is supereularian. Now we assume that \( D \) is strong and \( J_0 \) is a spanning subdigraph of \( D \) with at most two connected components. Then \( D/J_0 \) is strong with \(|V(D/J_0)| \leq 2 \). It follows that \( D/J_0 \) is hamiltonian, and so \( D \) is supereularian. Thus (v) follows.

Let \( T' \) be a trail of \( D[A(D) - A(J_0)] \) that intersects every component of \( J_0 \) with \( V(D) - V(J_0) \subseteq V(T') \), and let \( J_1, J_2, \ldots, J_c \) be the connected components of \( J_0 \). Since for each \( i \) with \( 1 \leq i \leq c \), \( V(T') \cap V(J_i) \neq \emptyset \) and so \( T = D[A(T') \cup A(J_0)] \) is connected. As \( V(D) - V(J_0) \subseteq V(T') \), \( T = D[A(T') \cup A(J_0)] \) is spanning in \( D \). Let \( v \in V(T) \). If \( v \in V(D) - V(T') \), we define \( d_{T'}^+(v) = d_{T'}^-(v) = 0 \). By (i), \( d_T^+(v) = d_T^-(v) + d_{T_0}^+(v) = d_{T_0}^-(v) = d_T^-(v) \), and so \( T \) is a spanning trail of \( D \). This justifies (vi).

To prove (vii), we assume that \( \lambda(D) \geq 2 \) and \( V(D - V(J_0)) = \{v_1, v_2, v_3\} \) such that \( G(D - V(J_0)) \) has a Hamilton cycle. Suppose first that \( D[\{v_1, v_2, v_3\}] \) is spanned by a 3-cycle. Then as \( D \) is strong, there must be arcs \((v', v^-), (v^+, v'') \in A(D)\) for some \( v', v'' \in \{v_1, v_2, v_3\} \) and \( v^-, v^+ \in V(J_0) \). It follows by Lemma 3.3(iv) or (vi) that \( D \) is supereularian. Hence we assume that \( D[\{v_1, v_2, v_3\}] \) does not contain a 3-cycle. Since \( D \) is a digraph, we may assume, by symmetry, that \((v_1, v_2), (v_2, v_3), (v_1, v_3) \in A(D)\) and \((v_3, v_1) \notin A(D)\). Since \( d_T^+(v_1) \geq \lambda(D) \geq 2 \), we must have \((v^+, v_1) \in A(D)\) for some \( v^+ \in V(J_0) \). Likewise, as \( d_T^+(v_3) \geq \lambda(D) \geq 2 \), we must have \((v_3, v^-) \in A(D)\) for some \( v^- \in V(J_0) \). It follows by Lemma 3.3(iv) that \( D \) is supereularian. This justifies (vii) and completes the proof of the lemma.

3.2 Structural properties

The rest of this section is devoted to the structural analysis for strong graphs whose arc-strong connectivity is at least as big as the matching number minus one. We start with a definition.
Definition 3.4 Let $M$ be a matching of $D$. For each $w \in V(D) - V(M)$, define

\[
\begin{align*}
M_{w}^{2,2} &= \{ e = [u_w(e), v_w(e)] \in M : |\{w, \{u_w(e), v_w(e)\}\}_{G(D)}| = 4 \}, \\
M_{w}^{2,1} &= \{ e = [u_w(e), v_w(e)] \in M : |\{w, \{u_w(e), v_w(e)\}\}_{G(D)}| = 3 \}, \\
M_{w}^{1,1} &= \{ e = [u_w(e), v_w(e)] \in M : |\{w, u_w(e)\}_{G(D)}| = |\{w, v_w(e)\}_{G(D)}| = 1 \}, \\
M_{w}^{1,0} &= \{ e = [u_w(e), v_w(e)] \in M : \\
&\quad \text{for some } v \in \{u_w(e), v_w(e)\}, |\{w, v\}_{G(D)}| = |\{w, \{u_w(e), v_w(e)\}\}_{G(D)}| = 2 \}, \\
M_{w}^{0,0} &= \{ e = [u_w(e), v_w(e)] \in M : |\{w, u_w(e)\}_{G(D)}| = |\{w, v_w(e)\}_{G(D)}| = 0 \}.
\end{align*}
\]

The following observation follows from Definition 3.4 and Theorem 2.52 (Berge Theorem).

Observation 3.5 Let $n = |V(D)|$ and $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ be a maximum matching of $D$. (i) As $M$ is a maximum matching, $V(D) - V(M)$ is a stable set. This implies that for any $w \in V(D) - V(M)$, $N_D(w) \subseteq V(M)$, and so by Definition 3.4, $d_D(w) = 4|M_{w}^{2,2}| + 3|M_{w}^{2,1}| + 2(|M_{w}^{1,1}| + |M_{w}^{1,0}|) + |M_{w}^{0,0}| = k$. (ii) Let $x, y \in V(D) - V(M)$ are distinct vertices, and $[u, v] \in M$. By Theorem 2.52, $D$ does not have an $M$-augmenting path, and so if $x \in N_D(u)$, then $y \notin N_D(v)$. (iii) As a consequence of (ii), if $x, y \in V(D) - V(M)$ are distinct vertices, then

\[
(M_{x}^{2,2} \cup M_{x}^{2,1} \cup M_{x}^{1,1}) \cap (M_{y}^{2,2} \cup M_{y}^{2,1} \cup M_{y}^{2,0} \cup M_{y}^{1,1} \cup M_{y}^{1,0}) = \emptyset.
\]

Throughout the rest of this section, we always assume that $D$ is a digraph with $k = \alpha'(D) \geq 3$, $n = |V(D)| \geq 2k + 3$, $J = J(D)$ is the symmetric core of $D$, and let $X = V(D) - V(M)$. For each $x \in X$, define

\[
k_1(x) = |M_{x}^{2,2}| + |M_{x}^{2,1}| + |M_{x}^{1,1}| \text{ and } k_2(x) = |M_{x}^{2,0}| + |M_{x}^{1,0}|.
\]

Lemma 3.6 Let $D$ be a digraph with $k = \alpha'(D) \geq 3$ and $\delta(D) \geq 2k - 2$, and $M$ be a maximum matching of $D$. If for some vertex $x_1 \in X$, both $d_D(x_1) \geq 2k - 1$ and $k_1(x_1) > 0$, then each of the following holds. (i) $k_1(x_1) = 1, k_2(x_1) \in \{k - 2, k - 1\}$, and for any vertex $x \in X - \{x_1\}$, $k_1(x) = 0$. (ii) $D$ has a stable set $\{v_1, v_2, ..., v_k\}$ such that $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ with $M_{x_1}^{2,2} \cup M_{x_1}^{2,1} \cup M_{x_1}^{1,1} = \{[u_1, v_1]\}$ and $\{u_1, u_2, ..., u_{k-1}, v_1\} \subseteq N_D(x_1) \subseteq \{u_1, u_2, ..., u_k, v_1\}$, and such that $J$ has a connected component $J'$ with $(X - \{x_1\}) \cup \{v_2, v_3, ..., v_k\} \subseteq V(J')$. (iii) $\{v_2, ..., v_k\} \subseteq V(J')$. Moreover, if $k \geq 4$, then $v_1$ lies in a nontrivial connected component of $J$. (iv) If $\lambda(D) \geq 2$, then $D$ is superregular. (v) If, in addition, $d_D(x_1) \geq 2k$, then either $(x_1, v_1), (v_1, x_1) \in A(D)$, or there exist at least $k - 1$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(x_1, u), (u, x_1) \in A(D)$.

Proof. Throughout the proof of this lemma, we let $k_1 = k_1(x_1)$ and $k_2 = k_2(x_1)$. Denote $M_{x_1}^{2,2} \cup M_{x_1}^{2,1} \cup M_{x_1}^{1,1} = \{[u_1, v_1], ..., [u_{k_1}, v_{k_1}]\}$ and $M_{x_1}^{2,0} \cup M_{x_1}^{1,0} = \{[u_{k_1+1}, v_{k_1+1}], ..., [u_{k_1+k_2}, v_{k_1+k_2}]\}$ with $\{u_{k_1+1}, ..., u_{k_1+k_2}\} \subseteq N_D(x_1)$.
Choose $x_2 \in X - \{x_1\}$ such that

$$k_1(x_2) = \max\{k_1(x) : x \in X - \{x_1\}\},$$ and let $k''_2 = \left| \bigcup_{j=1}^2 (M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \right|$. 

By Observation 3.5(i) and (iii),

$$2k - 1 \leq d_D(x_1) = 4|M_{x_1}^{2,2}| + 3|M_{x_1}^{2,1}| + 2(|M_{x_1}^{2,0}| + |M_{x_1}^{1,1}|) + |M_{x_1}^{1,0}| \leq 4k_1 + 2k_2,$$

$$2k - 2 \leq d_D(x_2) = 4|M_{x_2}^{2,2}| + 3|M_{x_2}^{2,1}| + 2(|M_{x_2}^{2,0}| + |M_{x_2}^{1,1}|) + |M_{x_2}^{1,0}| \leq 4k_1(x_2) + 2k''_2.$$ 

By adding the inequalities above side by side, and by Observation 3.5(iii), we have

$$4k - 3 \leq 4(k_1 + k_1(x_2) + k''_2) \leq 4k - 4(|M_{x_1}^{0,0}| + |M_{x_2}^{0,0}|).$$

It follows that $|M_{x_1}^{0,0}| + |M_{x_2}^{0,0}| = 0$. By Observation 3.5(iii),

$$\bigcup_{j=1}^2 (M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \subseteq M - \left( \bigcup_{j=1}^2 (M_{x_j}^{2,2} \cup M_{x_j}^{2,1} \cup M_{x_j}^{1,1}) \right),$$

and so by Observation 3.5(i) and by $k_1 > 0$, we have

$$N_D(x) \subseteq \bigcup_{j=1}^2 \left( V(M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \cap N_D(x_j) \right), \quad \text{for any } x \in X - \{x_1, x_2\}, \quad (5)$$

$$k_1(x_2) \geq k - (k_1 + k_1(x_2)) \geq \left| \bigcup_{j=1}^2 (M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \right|. \quad (6)$$

If $k_1 = 1$ and $k_1(x_2) = 0$, then as $d_D(x_1) \geq 2k - 1$, it would follow that $k_2 \in \{k - 2, k - 1\}$. Hence to prove Lemma 3.6(i), it suffices to show that $k_1 = 1$ and $k_1(x_2) = 0$. By contradiction, we assume that either $k_1 \geq 2$ or $k_1(x_2) > 0$. Then by (6), $k - 2 \geq \left| \bigcup_{j=1}^2 V(M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \right|$. Since $n = |V(D)| \geq 2k + 3$, there exists a vertex $x_3 \in X - \{x_1, x_2\}$. By $\delta(D) \geq 2k - 2$, (5) and by Observation 3.5(iii), $2(k - 1) \leq |N_D(x_3)| \leq 2\left| \bigcup_{j=1}^2 V(M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \right| \leq 2(k - 2)$, a contradiction. This proves that Lemma 3.6(i).

By (i), $k_1 = 1$. Let $[u_1, v_1]$ denote the only arc in $M_{x_1}^{2,2} \cup M_{x_1}^{2,1} \cup M_{x_1}^{1,1}$. As $k_2 \in \{k - 2, k - 1\}$, we can label the vertices and denote $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ such that $\{u_1, u_2, ..., u_{k-1}\} \subseteq N_D(x_1)$, and such that if $(X, \{u_k, v_k\}) \notin G(D)$, then $(X, \{u_k\}) \notin G(D)$. Hence $\{u_1, u_2, ..., u_{k-1}, v_1\} \subseteq N_D(x_1)$, and so by $\delta(D) \geq 2k - 2$, $N_D(x_1) = \{u_2, ..., u_k\}$. It follows by $\delta(D) \geq 2k - 2$ that $\{(u_j, x), (x, u_j) \in A(D)\}$ for any $2 \leq j \leq k$, and so $J$ has a connected component $J'$ containing the vertices $(X - \{x_1\}) \cup \{u_2, u_3, ..., u_k\}$. As $N_D(x_2) = \{u_2, u_3, ..., u_k\}$, $k \geq 3$ and $u_1, v_1 \in N_D(x_1)$, we conclude by Theorem 2.52 that $\{v_1, v_2, ..., v_k\}$ is a stable set of $D$ as any arc in $D$ incident with two distinct vertices in $\{v_1, v_2, ..., v_k\}$ would give rise to an $M$-augmenting path in $D$. This proves Lemma 3.6(ii).

For any $v_i$ with $2 \leq i \leq k$, as $\{v_1, v_2, ..., v_k\}$ is a stable set, $N_D(v_i) \subseteq V(D) - \{v_1, ..., v_k\}$. By Observation 3.5(iii) and by Lemma 3.6(ii), we further conclude that $N_D(v_i) \subseteq \{u_2, u_3, ..., u_k\}$. This, together with $\delta(D) \geq 2k - 2$, forces that $\{(u_j, v_i), (v_i, u_j)\} \subseteq A(D)$, for any $j$ with $2 \leq j \leq k$. Hence $\{v_2, ..., v_k\} \subseteq V(J')$. By Observation 3.5, $(X - \{x_1\}, \{v_1\}) \notin G(D)$, and so $N_D(v_1) \subseteq \{u_1, u_2, u_3, ..., u_k, x_1\}$. It follows that
\(|\{u_1, u_2, u_3, \ldots, u_k, x_1\}, \{v_1\}\rangle_{G(D)} \geq |N_D(v_1)| \geq 2k - 2\), and so there exist at least \((2k - 2) - (k + 1) \geq k - 3\) vertices \(z \in \{u_1, u_2, u_3, \ldots, u_k, x_1\}\) satisfying \((z, v_1), (v_1, z) \in A(D)\). Hence if \(k \geq 4\), then \(v_1\) lies in a non-trivial connected component of \(J\). This proves Lemma 3.6(iii).

Let \(J_0 = J[V(D) - \{u_1, v_1, x_1\}]\). By (ii) and (iii), \(J_0\) is a connected symmetric subdigraph of \(J\). As \([u_1, v_1], [v_1, x_1], [x_1, u_1] \in A(D)\), it follows by \(\lambda(D) \geq 2\) and Lemma 3.3(vii) that \(D\) is supereulerian. This proves (iv).

Finally, we assume that \(d_D(x_1) \geq 2k\) but \(|\{x_1\}, \{v_1\}\rangle_{G(D)} = 1\). Then \(|\{x_1\}, \{u_1, \ldots, u_k\}\rangle_{G(D)} \geq 2k - 1\), implying that there exist at least \(k - 1\) vertices \(u \in \{u_1, u_2, \ldots, u_k\}\) with \((x_1, u), (u, x_1) \in A(D)\). Hence (v) holds. This completes the proof of Lemma 3.6.

For a digraph \(D\) with vertex set \(V = V(D)\), recall \(D\) is a complete digraph if for any pair of distinct vertices \(u, v \in V\), \((u, v), (v, u) \in A(D)\). A complete digraph on \(n\) vertices will be denoted by \(K^*_n\). Define \(D_0\) to be the vertex disjoint union of three complete digraphs of order 3.

**Lemma 3.7** Let \(D\) be a digraph with \(k = \alpha'(D) \geq 3\), \(\delta(D) \geq 2k - 2\) and \(M\) be a maximum matching of \(D\). Suppose that \(\delta(D) \geq 2k - 2\) holds.

(i) If, for some vertex \(x_1 \in X\), \(d_D(x_1) \geq 2k - 1\) and \(k_1(x_1) = 0\), then for any \(x \in X\), \(k_1(x) = 0\).

(ii) If for some vertex \(x_1 \in X\), \(k_1(x_1) > 0\), then either \(D \cong D_0\), or \(k_1(x_1) = 1\) and \(k_1(x) = 0\) for any \(x \in X - \{x_1\}\).

**Proof.** Arguing by contradiction to prove (i), we may assume that \(x_2 \in X - \{x_1\}\) and \(k_1(x_2) > 0\). Let \([u_2, v_2] \in M_{x_2}^2 \cup M_{x_2}^1 \cup M_{x_2}^0\). Then by Observation 3.5(i), \(N_D(x_1) \subseteq V(M - \{u_2, v_2\})\). As \(d_D(x_1) \geq 2k - 1\), and as \(|M - \{u_2, v_2\}| = k - 1\), there exists an arc \([u_1, v_1] \in M - \{u_2, v_2\}\) such that \(|\{x_1\}, \{u_1, v_1\}\rangle_{G(D)} \geq 3.3.4\). Hence we must have \(k_1(x_1) > 0\), contrary to the assumption that \(k_1(x_1) = 0\). This proves Lemma 3.7(i).

Now assume that for some vertex \(x_1 \in X\), \(k_1(x_1) > 0\). Then there exists an arc \([u_1, v_1] \in M\) such that \(u_1, v_1 \in N_D(x_1)\). By Observation 3.5(ii), for any \(x \in X - \{x_1\}\), \(u_1, v_1 \notin N_D(x)\). Suppose that we have another vertex \(x_2 \in X - \{x_1\}\) with \(k_1(x_2) > 0\), or we have \(k_1(x) \geq 2\). Then there must be an arc \([u_2, v_2] \in M - \{u_1, v_1\}\) such that \(u_2, v_2 \in N_D(x_2)\) (if \(k_1(x_2) > 0\)), or \(u_2, v_2 \in N_D(x_1)\) (if \(k_1(x_1) \geq 2\)). If there exists a vertex \(x \in X\) with \(k_1(x) = 0\), then by \(d_D(x) \geq 2k - 2\), either \((x, \{u_1, v_1\}) \notin G(D)\) or \([x, \{u_2, v_2\}] \notin G(D)\). In either case, a contradiction to Observation 3.5(ii) is obtained. Thus, either \(k_1(x) > 0\) for any \(x \in X\), or \(k_1(x_1) = 1\) and \(k_1(x) = 0\) for any \(x \in X - \{x_1\}\).

To complete the proof of (ii), in the following we, assume that \(k_1(x) > 0\) for any \(x \in X\). If \(D \cong D_0\), then done. Hence we by contradiction assume that \(D \not\cong D_0\). Define \(S = \bigcup_{x \in X} (M_{x}^{2,0} \cup M_{x}^{1,0})\), \(m' = \min\{k_1(x) : x \in X\}\) and \(m'' = \sum_{x \in X, k_1(x) > 0}(k_1(x) - 1)\). Since \(k_1(x) > 0\) for any \(x \in X, m' > 0\). By Observation 3.5(iii), \(\bigcup_{x \in X} (M_{x}^{2,2} \cup M_{x}^{2,1} \cup M_{x}^{1,1}) \cup S\) is a disjoint union and is a subset of \(M\). This, together with \(|X| = n - 2k\), implies that

\[
k = |M| \geq \sum_{x \in X} k_1(x) + |S| = m'' + (n - 2k) + |S|.
\]

**Claim 1** *We have \(m'' = 0\), \(n = 2k + 3\), \(|X| = 3.*
By (7), \( k \geq m'(n - 2k) + |S| \). Let \( x' \in X \) satisfying \( k_1(x') = m' \). Then \( 4m' + 2|S| \geq d_D(x') \geq 2k - 2 \), and so \( |S| \geq k - 1 - 2m' \). Hence we have

\[
k \geq m'(n - 2k) + |S| \geq m'(n - 2k) + k - 1 - 2m' = m'(n - 2k - 2) + k - 1.
\]

(8)

With \( n \geq 2k + 3 \), (8) leads to the conclusion that \( 1 \geq m'(n - 2k - 2) \geq m' \geq 1 \), forcing \( m' = 1 \) and \( n = 2k + 3 \). Thus \( |X| = n - 2k = 3 \). By (7) and by \( |S| \geq k - 1 - 2m' = k - 3 \), we have \( k \geq m'' + 3 + (k - 3) = m'' + k \). This implies \( m'' = 0 \) and proves Claim 1.

By Claim 1, we may assume that \( X = \{x_1, x_2, x_3\} \). As \( m'' = 0 \), for any \( x \in X \), \( k_1(x) = 1 \). Fix an \( x_i \in X \) for \( 1 \leq i \leq 3 \). As \( k_1(x_i) = 1 \), we may assume that \( u_i, v_i \in N_D(x_i) \), and \( \{x_i, \{u_i, v_i\}\} \bigcap D = \emptyset \) for any \( j \) with \( j \neq i \). By Observation 3.5(ii), we observe that \( \{x_i, \{u_i, v_i\}\} \bigcap D = \emptyset \) for any \( 1 \leq i \leq 3 \) and \( h \neq i \). This implies that \( 4 + 2(k - 3) \geq |\{\{x_1, \{u_1, v_1\}\} \bigcap D| + \sum_{j=4}^{k} |\{x_j, \{u_j, v_j\}\} \bigcap D| = d_D(x_i) \geq 2k - 2 \), and so we must have \( d_D(x_i) = 2k - 2 \), \( \{\{x_1, \{u_1, v_1\}\} \bigcap D| = 4 \), and for \( j \) with \( 4 \leq j \leq k \), \( |\{x_j, \{u_j, v_j\}\} \bigcap D| = 2 \).

We further claim that \( \{v_1, ..., v_k\} \) is a stable set in \( D \). By contradiction, we assume that there exists an arc \( [v_i, v_j] \in A(D) \) for some \( 1 \leq i < j \leq k \). If \( j \leq 3 \), then \( \{x_1, u_i, [u_i, v_i], [v_i, v_j], [u_j, x_j]\} \) induces an \( M \)-augmenting path in \( D \). If \( i \leq 3 < j \), then choosing an index \( i' \neq i \) and \( 1 \leq i' \leq 3 \), \( \{x_i, u_i, [u_i, v_i], [v_i, v_j], [v_j, x_j]\} \) induces an \( M \)-augmenting path in \( D \). If \( i \geq 4 \), then \( \{x_1, u_i, [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_j]\} \) induces an \( M \)-augmenting path in \( D \). In any case, Theorem 2.52 is violated. Hence \( \{v_1, ..., v_k\} \) must be a stable set.

If \( k \geq 4 \), then \( N_D(v_4) \subseteq \{u_1, u_2, ..., u_k\} \). Since \( d_D(v_4) \geq 2k - 2 \), there must be an \( i \) with \( 1 \leq i \leq 3 \) such that \( [u_i, v_i] \in A(D) \). Pick \( i' \neq i \) and \( 1 \leq i' \leq 3 \). Then \( \{x_i, u_i, [u_i, v_i], [v_i, v_4], [v_4, u_4], [u_4, x_4]\} \) induces an \( M \)-augmenting path in \( D \), violating Theorem 2.52. Hence we must have \( k = 3 \). Recall that for each \( i \in \{1, 2, 3\} \), \( \{\{x_i, \{u_i, v_i\}\} \bigcap D| = 4 \). Since \( D \not\supseteq D_0 \) and \( d_D(u_i) \geq 2k - 2 = 4 \), we may assume that, either \( [u_i, v_i] \in A(D) \) or \( [u_i, u_j] \in A(D) \), for \( 1 \leq i, j \leq 3 \) with \( i \neq j \). Once again, \( \{x_i, v_i], [u_i, u_i], [u_i, v_i], [u_i, u_j], [u_j, v_j], [v_j, x_j]\} \) or \( \{x_i, v_i], [u_i, u_i], [u_i, v_i], [u_i, u_j], [u_j, v_j], [v_j, x_j]\} \) induces an \( M \)-augmenting path in \( D \). These contradictions indicate that if \( k_1(x) > 0 \) for any \( x \in X \), then we must have \( D \not\supseteq D_0 \). This proves Lemma 3.7(ii).

\[\text{Corollary 3.8} \quad \text{Let } k \geq 4 \text{ be an integer, } D \text{ be a digraph with } \lambda(D) \geq \alpha'(D) = k \text{ and } n = |V(D)| \geq 2k + 3. \text{ Then } J = J(D) \text{ is connected.}\]

\[\text{Lemma 3.9} \quad \text{Let } D \text{ be a digraph with } k = \alpha'(D) \geq 3 \text{ and } M \text{ be a maximum matching of } D. \text{ Suppose that for some vertex } x \in X, d_D(x) \geq 2k - 1 \text{ with } k_1(x) = 0. \text{ If } \delta(D) \geq 2k - 2, \text{ then there exists a labeling of the vertices of } V(M) \text{ such that } M = \{u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\} \text{ and each of the following holds.}\]
\[\text{(i) } N_D(x) = \{u_1, u_2, u_3, ..., u_k\}, \{x, [v_1, v_2, ..., v_k]\} \bigcap D = \emptyset \text{ and there exist at least } k - 1 \text{ vertices } u \in \{u_1, u_2, ..., u_k\} \text{ with } (x, u) \in A(D). \text{ Moreover, if } d_D(x) \geq 2k, \text{ then for any } u \in \{u_1, u_2, ..., u_k\}, \text{ we have } (x, u) \in A(D).\]
\[\text{(ii) For any } x \in X - \{x_1\}, N_D(x) \subseteq \{u_1, u_2, ..., u_k\}; \text{ and there exist at least } k - 2 \text{ vertices } u \in \{u_1, u_2, ..., u_k\} \text{ satisfying } (x, u) \in A(D).\]
\[\text{(iii) The vertex subset } \{v_1, v_2, ..., v_k\} \text{ is a stable set in } D. \text{ Furthermore, for each } v_j \text{ with } 1 \leq j \leq k, N_D(v_j) \subseteq \{u_1, u_2, ..., u_k\} \text{ and there exist at least } k - 2 \text{ vertices } u \in \{u_1, u_2, ..., u_k\} \text{ satisfying } (v_j, u) \in A(D).\]
\[\text{(iv) } J \text{ has at most two components; and if } \lambda(D) \geq 1, \text{ then } D \text{ is supereulerian.}\]
\textbf{Proof.} By Lemma 3.7(i), for any $x \in X$, $k_1(x) = 0$. By Observation 3.5(i), $N_D(x_1) \subseteq V(M)$. Hence by $d_D(x_1) \geq 2k - 1$ and $k(x_1) = 0$, we can label $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ so that $N_D(x_1) = \{u_1, u_2, u_3, \ldots, u_k\}$. Again by $d_D(x_1) \geq 2k - 1$, there must be at least $k - 1$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ satisfying $(x_1, u), (u, x_1) \in A(D)$. Similarly, if $d_D(x_1) \geq 2k$, then for any $u \in \{u_1, u_2, \ldots, u_k\}$, we have $(x_1, u), (u, x_1) \in A(D)$. It follows by $N_D(x_1) = \{u_1, u_2, u_3, \ldots, u_k\}$ and by Observation 3.5 that $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)} = \emptyset$. This verifies Lemma 3.9(i).

By (i), $N_D(x_1) = \{u_1, u_2, u_3, \ldots, u_k\}$. For any $x \in X - \{x_1\}$, by Observation 3.5(i) and (ii), $N_D(x) \subseteq \{u_1, u_2, \ldots, u_k\}$. By $\delta(D) \geq 2k - 2$, $d_D(x) \geq 2k - 2$, and so there must be at least $k - 2$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ with $(x, u), (u, x) \in A(D)$. This proves Lemma 3.9(ii).

To prove (iii), we argue by contradiction and assume that for some $1 \leq i < j \leq k$, an arc $[v_i, v_j]$ is in $A(D)$. Since $n \geq 2k + 3$, there exists a vertex $x_2 \in X - \{x_1\}$. By Lemma 3.9(ii), $N_D(x_2) \subseteq \{u_1, u_2, \ldots, u_k\}$. As $d_D(x_2) \geq 2k - 2$, we may assume that $u_i \in N_D(x_2)$, and so $\{[x_2, u_i], [u_i, v_j], [v_j, v_j], [v_j, u_i], [u_i, x_1]\}$ induced an $M$-augmenting path in $D$, contrary to Theorem 2.52. Hence $\{v_1, v_2, \ldots, v_k\}$ must be a stable set in $D$. Likewise, by Lemma 3.9(i) and (ii), and arc in $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)}$ will give rise to an $M$-augmenting path, contrary to Theorem 2.52. Thus $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)} = \emptyset$. Consequently, for each $v_j$ with $1 \leq j \leq k$, $N_D(v_j) \subseteq \{u_1, u_2, \ldots, u_k\}$. By $d_D(v_j) \geq 2k - 2$, there exist at least $k - 2$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ satisfying $(v_j, u), (u, v_j) \in A(D)$.

To show (iv), we first assume by (i) and by symmetry that for any $i$ with $1 \leq i \leq k - 1$, $(x_1, u_i)$ is a symmetric arc in $D$ and $[x_1, u_i] \in A(D)$. Thus $J$ has a connected component of $J'$ with $\{x_1, u_1, \ldots, u_{k-1}\} \subseteq V(J')$. Let $J''$ denote the connected component of $J$ with $u_k \in V(J'')$. As $k \geq 3$, it follows by (ii) that, for every $x \in X - \{x_1\}$, either $x \in V(J')$ or $x \in V(J'')$. Similarly, by (ii), for every $v \in \{v_1, v_2, \ldots, v_k\}$, either $v \in V(J')$ or $v \in V(J'')$. Hence $J$ has at most two connected components $J'$ and $J''$. It now follows by Lemma 3.3(v) that if $J$ is strong, then $J$ must be superconnected. This completes the proof of the lemma.

\textbf{Lemma 3.10} Let $D$ be a digraph with $k = \alpha'(D) \geq 3$, $\delta(D) \geq 2k - 2$ and let $M$ be a maximum matching of $D$ and $J = D(M)$ be the symmetric core of $D$. If for any $x \in X$, $k_1(x) = 0$, and if there exists an arc $e \in M$ with $(X, V(e))_{G(D)} = \emptyset$, then there exists a labeling of the vertices of $V(M)$ with $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ and $e = [u_k, v_k]$ such that each of the following holds.

(i) $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)} = \emptyset$, $(x_1, v_2, \ldots, v_{k-1})$ is a stable set in $D$ and $J$ has a connected component $J'$ with $X \cup \{u_1, u_2, \ldots, u_{k-1}\} \subseteq V(J')$.

(ii) If $\{v_1, v_2, \ldots, v_k\}$ is a stable set in $D$, then for any $j \in \{1, 2, \ldots, k\}$, there exist $k - 2$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ with $(v_j, u), (u, v_j) \in A(D)$, and $J$ has at most two connected components.

(iii) Suppose that $\{v_1, v_2, \ldots, v_k\}$ is not a stable set in $D$ and $[v_{k-1}, v_k] \in A(D)$. Then $(u_k, \{v_1, \ldots, v_{k-2}\})_{G(D)} = \emptyset$. Moreover, if $k \geq 4$, then $\{v_1, \ldots, v_{k-2}\} \subseteq V(J')$; and if $\lambda(D) \geq 2$, then $D$ is superconnected.

\textbf{Proof.} By Observation 3.5(i), for any $x \in X$, $N_D(x) \subseteq V(M)$. As for some $e \in M$, we have $(X, V(e))_{G(D)} = \emptyset$. By $k_1(x) = 0$, and $d_D(x) \geq 2k - 2$, we can label $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ with $e = [u_k, v_k]$ such that for any $x \in X$, $N_D(x) = \{u_1, u_2, \ldots, u_{k-1}\}$, and for any $i$ with $1 \leq i \leq k - 1$, $(x, u_i), (u_i, x) \in A(D)$. As $k \geq 3$ and $|X| = n - 2k \geq 3$, it follows that $J$ has a connected component $J'$ with $X \cup \{u_1, u_2, \ldots, u_{k-1}\} \subseteq V(J')$. As $k_1(x) = 0$ for any $x \in X$, we conclude that $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)} = \emptyset$.

We argue by contradiction to show that $\{v_1, v_2, \ldots, v_{k-1}\}$ is a stable set in $D$. Suppose that for some $1 \leq i < j \leq k - 1$, $(v_i, v_j) \in A(D)$. As $n - 2k \geq 3$, $D\{(x_1, u_i), [u_i, v_j], [v_j, v_j], [v_j, u_i], [u_i, x_1]\}$ is an
$M$-augmenting path, contrary to Theorem 2.52. This proves (i).

In the proof of (ii) and (iii), we let $J^2$, $J^3$ and $J^4$ be connected components of $J$ such that $u_k \in V(J^2)$,
$v_k \in V(J^3)$ and $v_{k-1} \in V(J^4)$.

Assume that \{v_1, v_2, ..., v_k\} is a stable set in $D$. Fix an arbitrary vertex $v_j$ with $1 \leq j \leq k$. By (i),
we have $N_D(v_j) \subseteq \{u_1, u_2, ..., u_{k-1}, u_k\}$, and so by $\delta(D) \geq 2k-2$, there must be at least $k-2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(v_j, u), (u, v_j) \in A(D)$. It follows by $k \geq 3$ and by (i) that either $v_j \in V(J^4)$ (if $u \neq u_k$) or $v_j \in V(J^2)$ (if $u = u_k$). Hence every vertex in $D$ is either in $J'$ or in $J^2$, and so $J$ has at most two connected components. This proves (ii).

To prove (iii), we assume by symmetry that \{v_{k-1}, v_k\} $\notin A(D)$. Fix a vertex $v_j$ with $1 \leq j \leq k-2$. If
$\{u_k, v_j\} \notin A(D)$, then by (i) and by $n \geq 2k+3$, $D[\{[x_1, u_j], [u_j, v_j], [v_j, u_k], [u_k, v_k], [u_k, v_{k-1}], [v_{k-1}, u_{k-1}], [v_{k-1}, x_2]\}]$ is an $M$-augmenting path, contrary to Theorem 2.52. Hence $(u_k, v_j) \notin A(D) = \emptyset$. This proves that
$\{u_k, v_1, ..., v_{k-2}\} \notin A(D)$, and so $N_D(v_1) \subseteq \{u_1, ..., u_{k-1}, v_k\}$. By $d_G(v_j) \geq 2k-2$, there exist at least
$k-2$ vertices $u' \in \{u_1, ..., u_{k-1}, v_k\}$ such that $(u', v_j), (v_j, u') \in A(D)$ if $k \geq 4$ then $u' \in \{u_1, ..., u_{k-1}\} \subseteq V(J^4)$, and so $v_j \in V(J^4)$. Thus \{v_1, ..., v_{k-2}\} $\subseteq V(J^4)$.

In the following, we assume that $\lambda(D) \geq 2$ to prove the following claim, which completes the proof of the lemma.

**Claim 2** Under the assumption of Lemma 3.10(iii), if $\lambda(D) \geq 2$, then each of the following holds.

(a) If $k \geq 5$, then $J$ has at most two components, and so by Lemma 3.3(v), $D$ is supereulerian.

(b) If $[u_k, v_{k-1}] \in A(D)$, then \{u_k, v_{k-1}\} $\notin A(D)$.

(c) If $k = 4$, then $J$ has at most two components, and so by Lemma 3.3(v), $D$ is supereulerian.

(d) If $k = 3$, then $J$ has a symmetric subgraph $J_0$ such that $G(D - V(J_0))$ is spanned by a 3-cycle, and so by Lemma 3.3(vii), $D$ is supereulerian.

Assume that $k \geq 5$. If $J^2 \neq J^3 = J^4$, then $J$ has at most two components. Hence we assume that either $J^2 \neq J^3$, whence $|\{u_k, v_{k-1}\}| \leq 1$; or $J^2 \neq J^4$, whence $|\{u_k, v_{k-1}\}| \leq 1$. Since $\{u_k, v_1, ..., v_{k-2}\} \notin A(D)$ and $(X, \{v_k, v_{k-1}\}) \notin A(D)$, we have $N_D(u_k) \subseteq \{u_1, ..., u_{k-1}, v_k\}$. This, together with $d_G(u_k) \geq 2k-2$, implies that $d_G(u_k) \geq 2k-2$, and so there exists at least $k-4$ vertices $u'' \in \{u_1, ..., u_{k-1}\}$ such that $(u_k, u''), (u'', u_k) \in A(D)$ for $k \geq 5$, $u_k \in V(J^4)$. Similarly, by (i), $N_D(v_{k-1}) \subseteq \{u_1, ..., u_{k-1}, v_k\}$ and so $|\{v_{k-1}, u_1, ..., u_{k-1}, u_k\}| \geq 2k-4$. Again by $k \geq 5$, there exists at least $k-4$ vertices $u'' \in \{u_1, ..., u_{k-1}, u_k\}$ such that $(v_{k-1}, u''), (u'', v_{k-1}) \in A(D)$, and so $v_{k-1} \in V(J^4)$. This indicates that $V(D) - V(J^4) \subseteq \{v_k\}$, and so Claim 2(a) follows.

By contradiction, we assume that \{u_k, v_{k-1}\} $\notin A(D)$ for some $j \in \{1, 2, ..., k-2\}$. Then \{v_1, u_1, u_j, v_j, v_k, v_{k-1}, u_k, v_{k-1}, v_k\} $\notin A(D)$, and so $\lambda(D) \geq 2$.

Assume that $k = 4$. Then $v_1, v_2 \in V(J^4)$ and $\{u_k, v_{k-1}\} \notin A(D)$ $\Rightarrow$ $N_D(u_k) \subseteq \{u_1, u_2, u_3, v_3\}$. Since $d_G(u_4) \geq 6$, for some $w \in \{u_1, u_2, u_3, v_3\}$, both $(w, u_k), (u_k, w) \in A(D)$. Hence either $J^2 = J^4$ (if $w \in \{u_1, u_2, u_3\}$), or $J^2 = J^3$ (if $w = v_3$), or $J^2 = J^4$ (if $w = v_3$), and so $J$ has at most three connected components $J^*, J^3$ and $J^4$. Similarly, $N_D(v_3) \subseteq \{u_1, u_2, u_3, v_4\}$ and $J^2 = J^4$ (if $w \in \{u_1, u_2, u_3, v_4\}$, both $(w', v_3), (v_3, w') \in A(D)$). Hence either $J^2 = J^3 = J^4$, or $J^2 = J^3$ with $V(J^4) \cap (V(J^3) \cup V(J^3)) = \emptyset$. It follows that either $J$ has at most two connected components
$J'$ and $J^2 = J^4$ and $J$ has at most three connected components $J'$, $J^3$ and $J^4$. When $J^2 = J^4$, we have $[u_4, v_3] \in A(D)$, and so by (b), $N_D(v_3) \subseteq \{u_1, u_2, u_4, u_4, v_3\}$. By $d_D(v_4) \geq 6$, we must have $J^3 = J'$ or $J^3 = J^4$ and so $J$ has at most two connected components $J'$ and $J^4$. This proves (c).

We now assume that $k = 3$. Assume first that $(u_3, v_2)_{G(D)} = \emptyset$. Then for each $z \in \{v_1, v_2, u_3\}$, as $N_D(z) \subseteq \{u_1, u_2, v_3\}$, $z \in V(J')$ or $z \in V(J^3)$. Hence $J$ has at most two connected components $J'$ and $J^3$, and so by Lemma 3.3(v), $D$ is supereulerian. Therefore, we assume that $[u_3, v_2] \in A(D)$. By (b), $|\{(v_1), \{v_2\}\}_{G(D)}| = 0$. By (i), $|\{(v_1), \{v_2\}\}_{G(D)}| = 0$. Hence $N_D(v_1) \subseteq \{u_1, u_2\}$. By $d_D(v_1) \geq 4$, $(v_1, u_1), (u_1, v_1) \in A(D)$, and so $v_1 \in V(J')$. Let $J_0 = J'[V(D) - \{u_3, v_2, v_3\}]$. As $[u_3, v_2], [v_2, v_3], [v_3, v_3] \in A(D)$, it follows from $\lambda(D) \geq 2$ and Lemma 3.3(vii) that $D$ is supereulerian. This completes the justification of Claim 2.

Lemma 3.11 Let $D$ be a digraph with $k = \alpha'(D) \geq 3$ and $\delta(D) \geq 2k - 2$, and $M$ be a maximum matching of $D$. If for any $x \in X$, $k_1(x) = 0$ and for any arc $e \in M$, $(X, V(e))_{G(D)} \neq \emptyset$, then there exists a labeling of the vertices of $V(M)$ such that $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$, $N_D(X) = \{u_1, u_2, ..., u_k\}$, and each of the following holds.

(i) $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$, and for any $x \in X$, there exists $k - 2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(x, u), (u, x) \in A(D)$.

(ii) $[v_1, v_2, ..., v_k]$ is a stable set in $D$, and for any $v_j$ with $1 \leq j \leq k$, there exist at least $k - 2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(u, v_j), (v_j, u) \in A(D)$.

(iii) If $\lambda(D) \geq 2$, then $D$ is supereulerian.

Proof. For any vertex $x \in X$, by Observation 3.5(i), $N_D(x) \subseteq V(M)$; by assumption, $k_1(x) = 0$ and

for any arc $e \in M$, $(X, V(e))_{G(D)} \neq \emptyset$. (9)

This, together with Observation 3.5(ii), implies that every arc in $M$ has exactly one vertex in $N_D(X)$. Thus we can denote $V(M) \cap N_D(X) = \{u_1, u_2, ..., u_k\}$ and $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$. This labeling of vertices in $V(M)$ implies that $N_D(X) \subseteq \{u_1, u_2, ..., u_k\}$, and so $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$. Fix an $x \in X$. Since $d_D(x) \geq 2k - 2$, for at least $k - 2$ vertices $u \in \{u_1, u_2, ..., u_k\}$, both $(u, x)$ and $(x, u)$ are in $A(D)$. Thus (i) holds.

By contradiction, assume that $\{v_1, v_2, ..., v_k\}$ is not a stable set in $D$. By symmetry, we may assume that $[v_1, v_2] \in A(D)$. For $i$ with $1 \leq i \leq k$, let $X_i = X \cap N_D(u_i)$. By (9), $X_i \neq \emptyset$, and so there exists a vertex $x_1 \in X_1$. If there exists a vertex $x_2 \in X_2 - \{x_1\}$, then $D([x_1, u_1], [u_1, v_1], [v_1, v_2], [v_2, u_2], [u_2, x_2])$ is an $M$-augmenting path, contrary to Theorem 2.52. Hence $X_2 = \{x_1\}$. By the same argument, we conclude that $X_1 = X_2 = \{x_1\}$. Since $n \geq 2k + 3$, we have $|X| \geq 3$, and so $X - \{x_1\} \neq \emptyset$. For any vertex $x \in X - \{x_1\}$, as $N_D(X) \subseteq \{u_1, u_2, ..., u_k\}$ and $X_1 = X_2 = \{x_1\}$, we conclude that $N_D(x) \subseteq \{u_3, u_4, ..., u_k\}$, which implies that $2k - 2 = 2\lambda(D) \leq d_D(x) \leq 2(k - 2)$, a contradiction. Thus $\{v_1, v_2, ..., v_k\}$ must be a stable set in $D$.

Fix a vertex $v_j$ with $1 \leq j \leq k$. By (i), $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$. As $\{v_1, v_2, ..., v_k\}$ is a stable set, we must have $N_D(v_j) \subseteq \{u_1, u_2, ..., u_k\}$. Since $\delta(D) \geq 2k - 2$, there exist at least $k - 2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(u, v_j), (v_j, u) \in A(D)$. This proves (ii).

We now assume that $\lambda(D) \geq 2$. By contradiction, we assume that $D$ is not supereulerian. Pick a vertex $x_1 \in X$ and let $J_1$ be the connected component of $J$ with $x_1 \in V(J_1)$. By (i), we may assume that
$u_1, \ldots, u_{k-2} \in V(J_1)$. Let $J_2$ and $J_3$ be connected components of $J$ with $u_{k-1} \in V(J_2)$ and $u_k \in V(J_3)$. By (i) and (ii), and by $k \geq 3$, for every vertex $v \in X \cup \{v_1, v_2, \ldots, v_k\}$, there exists an $i \in \{1, 2, 3\}$ such that $v \in V(J_i)$. It follows that $J$ has at most three connected components $J_1, J_2$ and $J_3$. By Lemma 3.3(v), if $J$ has at most two connected components, then $D$ is supereulerian. Hence $J$ must have exactly three components $J_1, J_2$ and $J_3$.

**Case 1** $k \geq 4$.

If there exists a vertex $v \in X \cup \{v_1, v_2, \ldots, v_k\}$ such that for distinct $i, j \in \{1, 2, 3\}$, $v \in V(J_i) \cup V(J_j)$, then as $k-2 \geq 2$, we have either $J_1 = J_2$, or $J_1 = J_3$, or $J_2 = J_3$, contrary to the assumption that $J$ has exactly three components. Therefore, for any $k \geq 4$, we have

$$V(J_1) = V(D) - \{u_{k-1}, u_k\}, \quad V(J_2) = \{u_{k-1}\} \quad \text{and} \quad V(J_3) = \{u_k\}. \quad (10)$$

Thus for any $x \in X$, and $u \in \{u_1, \ldots, u_{k-2}\}$ and any $v \in \{v_1, v_2, \ldots, v_k\}$, the arcs $(x, u), (u, v)$ are symmetric in $D$. As $\delta(D) \geq 2k-2$, we conclude that for any $v \in X \cup \{v_1, v_2, \ldots, v_k\}$, $d_D(v) = 2k-2$ and $|(v, u_k)| = |(v, u_k)|$. If $[u_{k-1}, u_k] \in A(D)$, then by $\lambda(D) > 0$ and by Lemma 3.3(iv), $D$ is supereulerian. Thus $(u_{k-1}, u_k)$ is a symmetric arc of $D$. If $D - A(J_1)$ has a cycle $C$ containing both $u_{k-1}$ and $u_k$, then $D[A(J_1) \cup D(C)]$ is a spanning closed trail of $D$, and so $D$ is supereulerian. Hence we assume $D - A(J_1)$ does not have a cycle or disjoint cycles containing both $u_{k-1}$ and $u_k$.

Since $\lambda(D) \geq 2$, there exist vertices $v^-, w^-, w^+, w^+ \in V(J_1)$ such that

$$(v^-, u_{k-1}), (w^-, u_k), (u_{k-1}, v^+), (u_k, w^+) \in A(D). \quad (11)$$

Since $J_1, J_2$ and $J_3$ are distinct components of $J$, thus, we assume that $v^+ \neq w^+$ and $v^- \neq w^-$. If $v^-, w^+ \in X \cup \{v_1, \ldots, v_k\}$, then $(w^-, u_1), (u_1, w^+), (v^-, u_1) \in A(J_1)$. Let $J_1' = J_1 - \{(w^+, u_1), (u_1, w^+), (v^-, u_1)\}$. As $|X| \geq 3$ and $k \geq 4$, $J_1'$ is a connected symmetric subdigraph of $D$, and by (11), $D - A(J_1')$ has a trail $w^- u_k w^+ u_1 v^- u_{k-1} v^+$. By Lemma 3.3(iv) with $J' = J_1'$, $D$ is supereulerian.

Suppose that $|\{u_1, \ldots, u_{k-2}\} \cap \{v^-, w^+\}| = 1$ and $|(X \cup \{v_1, \ldots, v_k\}) \cap \{v^-, w^+\}| = 1$. By symmetry, we assume that $v^- = u_1$ and $w^+ \in X \cup \{v_1, \ldots, v_k\}$. As $(w^+, u_1) \in A(J_1)$ is symmetric arcs of $D$. Let $J_2' = J_1 - \{(w^+, u_1), (u_1, w^+)\}$. As $|X| \geq 3$ and $k \geq 4$, $J_2'$ is a connected symmetric subdigraph of $D$, and by (11), $D - A(J_2')$ has a trail $w^- u_k w^+ u_1 u_{k-1} v^+$. It follows from Lemma 3.3(iv) with $J' = J_2'$ that $D$ is supereulerian. Hence we may assume that $v^- = u_1$ and $w^+ \in \{u_1, \ldots, u_{k-2}\}$. By (10), $(w^+, x_1), (x_1, v^-) \in A(J_1)$ are symmetric arcs of $D$. As $|X| \geq 3$ and $k \geq 4$, $J_1 - x_1$ is a connected symmetric subdigraph of $D$, and by (11), $D - A(J_1 - x_1)$ has a trail $w^- u_k w^+ x_1 v^- u_{k-1} v^+$. By Lemma 3.3(iv) with $J' = J_1 - x_1$, $D$ is supereulerian.

**Case 2** $k = 3$.

By definition, for each $i \in \{1, 2, 3\}$, $v_i \in V(J_i)$. By relabeling the vertices $u_1, u_2$ and $u_3$, we assume that $u_i \in V(J_i)$. By (ii) and by $\delta(D) \geq 4$, every $v_i$ is adjacent to a $u_j$ by a pair of symmetric arcs. Therefore, we may relabel $v_1, v_2, v_3$ and assume that $(u_i, v_i) \in A(J_i)$ is a symmetric arc of $D$. 37
Let $D' = D/J$, and denote $V(D') = \{ z_1, z_2, z_3 \}$, where $z_i \in V(D')$ be the vertex onto which $J_i$ is contracted. If $D'$ has a Hamilton cycle, then by Lemma 3.3(v), $D$ is superuniversal. Hence we may assume that $D$ is not Hamiltonian. By (i), (ii), $\lambda(D) \geq 2$, and the fact that for $i \in \{1, 2, 3 \}$, $d_D(v_i) = 4$, we observe that

$$
(i' \text{ or } i'' \text{ or } i''') = \{1, 2, 3 \}, \text{ then } |(v_{i'}, \{ u_{i''}, u_{i'''} \})| = 1 \text{ and } |(\{ u_{i''}, u_{i'''} \}, v_j)D| = 1.
$$

(12)

By (12) and by symmetry, we assume that $(v_1, u_2), (v_3, v_1) \in A(D)$. Thus $(z_1, z_2), (z_2, z_1) \in A(D')$. As $D'$ is not Hamiltonian, we assume that $(z_2, z_3) \notin A(D')$. By (12) and since $(z_2, z_3) \notin A(D')$, we conclude that $(v_3, v_2), (v_2, v_3) \in A(D)$. These force, by (12), that $(v_2, u_1), (v_1, v_3) \in A(D)$. As $(v_1, v_3), (v_3, u_2), (v_2, u_1) \in A(D)$, it follows that $D'$ must be hamiltonian, a contradiction. This proves that in Case 2, $D$ is also superuniversal. This completes the proof of the lemma.

**Lemma 3.12** Let $k \geq 3$ be an integer, $D$ be a digraph with $k = \alpha'(D) \geq 3$, $\delta(D) \geq 2k - 2$, and $M$ be a maximum matching of $D$. Suppose that for some $x \in X$, $k_1(x_1) > 0$. Then each of the following holds.

(i) Either $D \cong D_0$, or $J$ has a connected component $J'$ such that the subdigraph $D_1 = D - V(J')$ satisfies $|V(D_1)| \leq 3$ and that $G(D_1)$ is spanned by a $3$-cycle or a $K_2$.

(ii) If, in addition, $\lambda(D) \geq 2$, then $D$ is superuniversal.

**Proof.** As $k_1(x_1) > 0$, there exists an arc $e = [u_1, v_1] \in M$ with $u_1, v_1 \in N_D(x_1)$. By Lemma 3.7(ii), $D \cong D_0$, or $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$. Thus to prove (i), it suffices to assume that $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$ to show that the desired $J'$ and $D_1$ exist.

Fix a vertex $x \in X - \{x_1\}$. By Observation 3.5(ii), $N_D(x) \subseteq V(M) - \{u_1, v_1\}$; and by $k_1(x) = 0$, for any $e \in M$, $|N_D(x) \cap V(e)| \leq 1$. Hence we can label $M = \{ [u_1, v_1], [u_2, v_2], ..., [u_k, v_k] \}$ such that $N_D(x) \subseteq \{ u_2, ..., u_k \}$. By $\delta(D) \geq 2k - 2$, we conclude that for any $u_i$ with $2 \leq i \leq k$, $(x, u_i), (u_i, x) \in A(D)$. It follows that $J$ has a connected component $J'$ such that $(X - \{x_1\}) \cup \{ u_2, ..., u_k \} \subseteq V(J')$.

We claim that $\{ v_1, v_2, ..., v_k \}$ is a stale set. Assume by contradiction that for some $1 \leq i < j \leq k$, $[v_i, v_j] \in A(D)$. If $i = 1$, then $D([(x_1, u_1), [u_1, v_1], [v_1, v_j],[v_j, u_j],[u_j, x_2]])$ is an $M$-augmenting path; If $i > 1$, then $D([(x_2, u_2), [u_2, v_1], [v_1, v_j],[v_j, u_j],[u_j, x_3]])$ is an $M$-augmenting path. In either case, a contradiction to Theorem 2.52 is obtained. Hence $\{ v_1, v_2, ..., v_k \}$ is a stable set.

Fix a vertex $v_j$ with $2 \leq j \leq k$. If $[u_1, v_j] \in A(D)$, then $\{ [x_1, v_1], [v_1, u_1], [u_1, v_j], [v_j, u_j], [u_j, x_2] \}$ induces an $M$-augmenting path in $D$, contrary to Theorem 2.52. Hence $(u_1, \{ v_2, ..., v_k \})_{G(D)} = \emptyset$ and so $N_D(v_j) \subseteq \{ u_2, ..., u_k \}$. As $d_D(v_j) \geq 2k - 2$, we conclude that for any $u \in \{ u_2, ..., u_k \}$ with $(u, v_j), (v_j, u) \in A(D)$, and so $(X - \{x_1\}) \cup \{ u_2, ..., u_k \} \cup \{ v_2, ..., v_k \} \subseteq V(J')$. As $[x_1, u_1], [x_1, v_1], [u_1, v_1] \in A(D)$, Lemma 3.12(i) is justified.

By Lemma 3.12(ii) and since $\lambda(D) \geq 2$, we observe that $D \not\cong D_0$ and so $J(D)$ has a connected component $J'$ such that the subdigraph $D_1 = D - V(J')$ satisfies $|V(D_1)| \leq 3$ and that $G(D_1)$ is spanned by a $3$-cycle or a $K_2$. If $G(D_1)$ is spanned by a $3$-cycle, then by Lemma 3.3(vii), $D$ is superuniversal. If $G(D_1)$ is spanned by a $K_2$, then by Lemma 3.3(iv), $D$ is superuniversal. Hence Lemma 3.12(ii) holds.
3.3 Spanning trails in digraphs with small matching numbers

In this subsection, we will identify a family $D$ of digraphs, and use it to prove Theorem 3.1(i). Let $D$ be a digraph and let $X$ denote a set of arcs not in $A(D)$ satisfying $\bigcup_{e \in X} V(e) \subset V(D)$. Define $D + X$ to be the digraph with vertex set $V(D)$ and arc set $A(D) \cup X$. If $X \subset A(D)$ (or $X \subset V(D)$, respectively), then define $D - X = D[A(D) - X]$ (or $D - X = D[V(D) - X]$, respectively). We often use $D + e$ for $D + \{e\}$, $D - e$ for $D - \{e\}$ and $D - v$ for $D - \{v\}$. We start with some examples.

![Figure 15. Digraph family $D(t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3)$](image)

**Example 3.13** Let $n, t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3$ be nonnegative integers with $n = 2 + t_1 + t'_1 + t''_1 + t_2 + t'_2 + t''_2 + t_3$. Define mutually disjoint vertex sets $X, Y$ and $Z$ as follows,

\[
X = \{x_1, x_2, \ldots, x_{t_1}, x'_1, x''_1, x'_2, x''_2, \ldots, x'_{t_1}\}
\]
\[
Y = \{y_1, y_2, \ldots, y_{t_2}, y'_1, y''_1, y'_2, y''_2, \ldots, y'_{t_2}\}
\]
\[
Z = \{z_1, z_2, \ldots, z_{t_3}\}
\]

and $w_1, w_2$ be two vertices not in $X \cup Y \cup Z$; and define mutually disjoint arc sets $A_X, A_Y$ and $A_Z$ as

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follows,

\[
\begin{align*}
A_X &= \left( \bigcup_{i=1}^{t_1} \{(w_1, x_i), (x_i, w_2)\} \cup \left( \bigcup_{i=1}^{t'_1} \{(w_1, x'_i), (x'_i, w_1), (x'_i, w_2)\} \right) \\
A_Y &= \left( \bigcup_{i=1}^{t_2} \{(w_2, y_i), (y_i, w_1)\} \cup \left( \bigcup_{i=1}^{t'_2} \{(w_2, y'_i), (y'_i, w_2), (y'_i, y_i)\} \right) \\
A_Z &= \bigcup_{i=1}^{t_3} \{(w_1, z_i), (z_i, w_2), (z_i, z_2), (z_i, w_2)\}.
\end{align*}
\]

Define a digraph \( D = D(t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3) \) with \( V(D) = \{ w_1, w_2 \} \cup X \cup Y \cup Z \) and arc set \( A(D) = A_X \cup A_Y \cup A_Z \). (See Fig. 15.)

**Observation 3.14** Define a digraph \( D = D(t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3) \) with that \( n \geq 4 \) and \( \lambda(D) > 0 \). Then each of the following holds.

(i) \( D \) is supereulerian if and only if both \( t_1 \leq t_2 + t'_2 + t''_2 + t_3 \) and \( t_2 \leq t_1 + t'_1 + t''_1 + t_3 \).

(ii) \( D \) has a spanning trail if and only if one of the following holds.

\[
\begin{align*}
&\text{both } t_1 \leq t_2 + t'_2 + t''_2 + t_3 + 1 \text{ and } t_2 \leq t_1 + t'_1 + t''_1 + t_3; \\
&\text{both } t_1 \leq t_2 + t'_2 + t''_2 + t_3 \text{ and } t_2 \leq t_1 + t'_1 + t''_1 + t_3 + 1.
\end{align*}
\]

**Proof.** We are to justify the conclusions of Example 3.13. By inspection, the conclusions (i) and (ii) holds if \( n = 4 \). Thus we assume that \( n \geq 5 \). Let \( J = J(D) \) be the symmetric core of \( D \).

We assume that both \( t_1 \leq t_2 + t'_2 + t''_2 + t_3 \) and \( t_2 \leq t_1 + t'_1 + t''_1 + t_3 \) to show by induction on \( t_1 + t_2 \) that \( D \) is supereulerian. If \( t_1 + t_2 = 0 \), then \( J \) has at most two connected components, and so by Lemma 3.3(v), \( D \) is supereulerian. Assume that \( t_1 + t_2 > 0 \) and that for smaller values of \( t_1 + t_2 \), \( D \) is supereulerian. By symmetry, we may assume that \( t_1 \geq t_2 \), and so \( t_1 > 0 \). If \( t_2 > 0 \), then let \( D_1 = D - \{ x_1, y_1 \} \). Then as \( D_1 = D(t_1 - 1, t'_1, t''_1, t_2 - 1, t'_2, t''_2, t_3) \), by induction, \( D_1 \) has a spanning eulerian subdigraph \( H_1 \), and so \( D[A(H_1) \cup \{(w_1, x_1), (x_1, w_2), (w_2, y_1), (y_1, w_1)\}] \) is a spanning eulerian subdigraph of \( D \). Hence we assume that \( t_2 = 0 \). Since \( t_1 + t_2 + t'_2 + t''_2 + t_3 \), there exists a \( v \in \{ y'_1, y''_2, \ldots, y''_{t_2}, \} \) such that \( (w_2, v), (v, w_1) \in A(D) \). Let \( D_2 = D - \{ x_1, v \} \). By induction, \( D_2 \) has a spanning eulerian subdigraph \( H_2 \), and so \( D[A(H_2) \cup \{(w_1, x_1), (x_1, w_2), (w_2, v), (v, w_1)\}] \) is a spanning eulerian subdigraph of \( D \).

Conversely, we assume that \( D \) has a spanning eulerian subdigraph \( H \). We again argue by induction on \( t_1 + t_2 \) to show that both \( t_1 \leq t_2 + t'_2 + t''_2 + t_3 \) and \( t_2 \leq t_1 + t'_1 + t''_1 + t_3 \). As these inequalities holds when \( t_1 = t_2 = 0 \), we assume by symmetry, that \( t_1 \geq t_2 \) and \( t_1 > 0 \). If \( t_2 > 0 \), then \( (w_1, x_1), (x_1, w_2), (w_2, y_1), (y_1, w_1) \in A(H) \), and so \( H - \{ x_1, y_1 \} \) is a spanning eulerian subdigraph of
$D - \{x_1, y_1\}$, and so by induction. $t_1 - 1 \leq (t_2 - 1) + t_2' + t_3' + t_3$ and $t_2 - 1 \leq (t_1 - 1) + t_1' + t_1'' + t_3$. Hence we assume that $t_2 = 0$. As $H$ is a spanning eulerian subdigraph, there must be a $v \in \{y_1', y_2', \ldots, y_n', y_{n+1}', \ldots, y_{2n}', z_1, z_2, \ldots, z_{2n}\}$ such that $(w_2, v), (v, w_1) \in A(H)$. Let $H'$ denote the nontrivial component of $H - \{(w_1, x_1), (x_1, w_2), (w_2, v), (v, w_1)\}$ and $D'$ the nontrivial component of $D - \{(w_1, x_1), (x_1, w_2), (w_2, v), (v, w_1)\}$. Then $H'$ is a spanning eulerian subdigraph of $D'$, and so by induction, we have $t_2 = 0$ and $t_1 - 1 \leq t_1' + t_2' + t_3 - 1$. Hence (i) holds by induction.

To prove (ii), it suffices to investigate spanning trails in a nonsupereularian $D$. By (i), any strong digraph $D(0, t_1', t_2', 0, t_2', t_3)$ is supereularian, and so we assume that $\max\{t_1, t_2\} > 0$. We make the following claim.

**Claim 3** Let $D = D(t_1, t_1', t_2, t_2', t_3)$ with $\lambda(D) > 0$ be a non supereularian digraph. If $D$ has a spanning trail, then $D$ has a spanning $(u,v)$-trail $T$ satisfying

\[
\text{both } u \in \{x_1, x_2, \ldots, x_t\} \text{ and } v = w_2, \text{ or both } u \in \{y_1, y_2, \ldots, y_{2n}\} \text{ and } v = w_1. \tag{16}
\]

**Proof.** Since $D$ is not supereularian, by Observation 3.14(i), $\max\{t_1, t_2\} > 0$. We assume that $t_1 > 0$. Let $T'$ be a spanning $(u', v')$-trail of $D$. We construct a spanning trail satisfying (16) from the following cases.

We note that as $T'$ is a $(u', v')$-trail, we have

\[
d_{T'}^+(u') - d_{T'}^-(u') = 1 \quad \text{and} \quad d_{T'}^+(v') - d_{T'}^-(v') = 1. \tag{17}
\]

**Case 1** $\{u', v'\} = \{w_1, w_2\}$.

If $u' = v'$, then $D$ is supereularian, contrary to the assumption of Claim 3. If $T'$ is a $(w_1, w_2)$-trail and $d_{T'}^+(w_1) \geq 2$, then $T' - (w_1, x_1)$ is a spanning $(x_1, w_2)$-trail of $D$ satisfying (16). If $T'$ is $(w_1, w_2)$-trail and $d_{T'}^+(w_1) = 1$, then there exists a vertex $y \in X \cup Y \cup Z$ such that $(y, w_2) \in A(T')$ and $(y, w_1) \in A(D) - A(T')$, so $T' - (y, w_2) + (y, w_1)$ is an eulerian digraph of $D$, contrary the assumption of Claim 3. The proof for the case when both $T'$ is a $(w_2, w_1)$-trail and $t_2 > 0$ is similar so it is omitted. Hence we assume that $T'$ a $(w_2, w_1)$-trail and $t_2 = 0$. As $t_1 > 0$, $(w_1, x_1), x_1, w_2 \in A(T')$. Since $n \geq 4$ and $T'$ is a spanning closed trail in $D$, there must be a vertex $y \in V(D)$ such that $(w_2, y), (y, w_1) \in A(T')$. It follows that $y \in Y \cup Z$ and $T' - y$ is an eulerian subdigraph of $D$. Since $t_2 = 0$, we have $y \in \{y_1, y_2, \ldots, y_n', y_{n+1}', \ldots, y_{2n}'\} \cup Z$, and so $y$ is incident with a pair of symmetric arcs $(y, w), (w, y)$ for some $w \in \{w_1, w_2\}$. It follows that $(T' - y) + \{(y, w), (w, y)\}$ is a spanning closed trail of $D$, contrary the assumption of Claim 3.

**Case 2** Both $u' \in \{w_1, w_2\}$ and $v' \in X \cup Y \cup Z$, or both $u' \in X \cup Y \cup Z$ and $v' \in \{w_1, w_2\}$.

Suppose first that $u' \in \{w_1, w_2\}$ and $v' \in X \cup Y \cup Z$. If $d_{T'}^+(v') = 1$, then by (17)and by (10), for some for some $i \in \{1, 2\}$, $(v', w_i) \notin A(D) - A(T')$. It follows that $T' + (v', w_i)$ is a spanning $(u', w_i)$-trail. By Case 3, we are done. Hence we assume that $d_{T'}^+(v') = 2$. Then by (17) and by (13), for some $i \in \{1, 2\}$, $(w_3, v'), (w_3, v'), (v', v_i') \in A(T')$. It follows that $T' - (w_3, v')$ is a spanning $(u', v_i')$-trail. By Case 3, we are done. The proof for the case when both $u' \in X \cup Y \cup Z$ and $v' \in \{w_1, w_2\}$ is similar and so it is omitted.

**Case 3** $u', v' \in X \cup Y \cup Z$.

By (17), either $d_{T'}^+(u') = 1$ and for some $j_1 \in \{1, 2\}$, $(w_{j_1}, u') \in A(D) - A(T')$, or $d_{T'}^+(u') = 2$ and for some $j_2 \in \{1, 2\}$, $(u', w_1), (u', w_2), (w_{j_2}, u') \in A(T')$. Likewise, either $d_{T'}^+(v') = 1$ and for some $j_3 \in$
\( \{1, 2\}, (v', w_{j3}) \in A(D) - A(T'), \) or \( d^-_{T'} (v') = 2 \) and for some \( j_4 \in \{1, 2\} \), \((w_1, v'), (w_2, v'), (v', w_{j4}) \in A(T') \). It follows that

\[
T'' = \begin{cases} 
T' + \{(w_{j3}, v') \}, (v', w_{j3}) \} & \text{if } d^+_{T'} (v') = 1 \text{ and } d^-_{T'} (v') = 1, \\
(T' - \{(w_{j3}, v') \}) + \{(v', w_{j3}) \} & \text{if } d^+_{T'} (v') = 2 \text{ and } d^-_{T'} (v') = 1, \\
(T' - \{(w_{j3}, v') \}) + \{(v', w_{j3}) \} & \text{if } d^+_{T'} (v') = 1 \text{ and } d^-_{T'} (v') = 2, \\
(T' - \{(w', w_{j3}) \}, (w_{j3}, v') \} & \text{if } d^+_{T'} (v') = 2 \text{ and } d^-_{T'} (v') = 2,
\end{cases}
\]

is a spanning \((w', w'')\)-trail of \( D \), for some \( w', w'' \in \{w_1, w_2\} \). By Case 3.3, we are done.

Assume that (14) holds. Then \( t_1 \geq 1 \) and so \( D - \{x_1\} \) satisfies the inequalities in Observation 3.14(i). By the definition of \( D \) in Observation 3.14, \( \lambda(D - \{x_1\}) > 0 \) if and only if either \( t_3 > 0 \), or both \( (t_1 - 1) + t'_1 + t''_1 > 0 \) and \( t_2 + t'_2 + t''_2 > 0 \). As \( \lambda(D) > 0 \), if \( t_3 = 0 \), then \( t_2 + t'_2 + t''_2 > 0 \). Therefore, if \( \lambda(D - \{x_1\}) = 0 \), then \( t_2 > 0 \) and \( t'_2 + t''_2 > 0 \), and so by (14), we must have \( t_1 = 1 \) and \( t'_1 + t''_1 = 0 \). These, together with (14), imply that \( D \) itself satisfies the inequalities in Observation 3.14(i), and so \( D \) is supersolvable, a contradiction. Hence we must have \( \lambda(D - \{x_1\}) > 0 \). By Observation 3.14(i), \( D - \{x_1\} \) has a spanning closed trail \( Q \). It follows that \( Q + \{(x_1, w_2)\} \) is a spanning \((x_1, w_2)\)-trail of \( D \). With a similar argument, if (15) holds, then \( D \) also has a spanning trail.

Conversely, assume that \( D \) has a spanning trail. If \( D \) has a spanning closed trail, then by Observation 3.14(i), each of (14) and (15) is satisfied. Hence we assume that \( D \) is not supersolvable. By Claim 3, we assume by symmetry that \( D \) has a spanning \((x_1, w_2)\)-trail. Then \( D - x_1 \) has a spanning closed trail, and so (14) follows from Observation 3.14(i).

**Definition 3.15** Using the notation used in Observation 3.14, we introduce a digraph family \( D(n) \) for each \( n \geq 4 \). Define a digraph \( D \in D(n) \) if and only if each of the following holds.

(F1) \( D \) has a subdigraph \( D' \), (called the corresponding digraph of \( D \)), such that there exist nonnegative integers \( t_1, t'_1, t_2, t'_2, t_3 \) satisfying \( |V(D')| = 2 + t_1 + t'_1 + t_2 + t'_2 + t_3 \geq 4 \) and \( D' = D(t_1, t'_1, t_2, t'_2, t_3) \) (as defined in Observation 3.14) such that both (14) and (15) are violated.

(F2) For each \( i \in \{1, 2\} \), let \( s_i \) be a nonnegative integer and \( D_i \) be digraph with \( V(D_i) = \{(w_i, w^i_{j1}), \ldots, w_i^i_{j2}\} \) and \( A(D) = \{(w_i, w^i_j), (w^i_j, w_i) : 1 \leq j \leq s_i, \) such that \( V(D_1) \cap V(D_2) = \emptyset \) and \( V(D_1) \cap V(D_2) = \{w_i\} \). When \( s_i = 0 \), then \( D_i \) consists of a single vertex \( w_i \).

(F3) Define \( D \) to be the digraph with \( V(D) = V(D') \cup V(D_1) \cup V(D_2) \) and \( A(D) = A(D') \cup A(D_1) \cup A(D_2) \), and let \( n = |V(D)| \).

By Lemma 3.3(vii) and using the notation in Definition 3.15, a digraph \( D \in D(n) \) has a spanning trail if and only if the corresponding \( D' \) of \( D \) has a spanning trail. The following follows from Observation 3.14.

For any digraph \( D \in D(n) \), \( D \) does not have a spanning trail.

**Corollary 3.16** Let \( D \) be a digraph obtained from a digraph \( D' = D(t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3) \) (as defined in Observation 3.14) with \( 4 = |V(D')| = 2 + t_1 + t'_1 + t''_1 + t_2 + t'_2 + t''_2 + t_3 \) by attaching a number of 2-cycles to each vertex of \( V(D') \). Then \( D \) is supersolvable if and only if \( D' \) is strong.

**Proof.** By Lemma 3.3 (vii), it suffices to examine these properties for \( D' \). Since \( D \) is strong, by the way we form \( D \) from \( D' \), \( D' \) is also strong. By Example 3.13, \( D' \) is strong if and only if both \( t_1 + t'_1 + t''_1 + t_3 > 0 \)
and \( t_2 + t_2' + t_2'' + t_3 > 0 \). As \( t = t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3 \), we have both \( t_1 \leq t_2 + t_2' + t_2'' + t_3 \) and \( t_2 \leq t_1' + t_1'' + t_1 + t_3 \). Thus Corollary 3.16 follows from Observation 3.14(i).

\[ \Box \]

**Lemma 3.17** Let \( D \) be a digraph with \(|V(D)| = 5\) such that \( G(D) \) has a Hamilton cycle. If \( D \) is strongly connected, then \( D \) has a spanning trail.

**Proof.** If \( D \) is supereulerian, then \( D \) has a spanning trail. Hence we assume that \( D \) is not supereulerian to show that \( D \) has a spanning trail. Let \( c \) be the length of a longest cycle in \( D \). As \( D \) is not supereulerian, we have \( 3 \leq c \leq 4 \). Suppose first that \( c = 3 \). Let \( C \) be a 3-cycle with arcs \( A(C) = \{(z_1, z_2), (z_2, z_3), (z_3, z_1)\} \).

Fix a vertex \( x \in V(D) - V(C) \). Since \( D \) is strong, there exist vertices \( z_x', z_x'' \in \{z_1, z_2, z_3\} \) such that \( D \) contains a \((x, z_x')\)-path \( P_x'\) and a \((z_x'', x)\)-path \( P_x''\). If for any \( x \in V(D) - V(C) \), we always have \( z_x' = z_x'' \), then \( D \) would be supereulerian, a contradiction. Hence there exists a vertex \( x_1 \) such that \( z_{x_1}' \neq z_{x_1}'' \). By symmetry, we assume that \( z_2 = z_{x_1}' \) and \( z_3 = z_{x_1}'' \). Since \( c = 3 \), \( D \) does not have a 4-cycle and so we must have \((x_1, z_2), (z_3, x_1) \in A(D)\). Let \( x_2 \) denote the only vertex in \( V(D) - \{z_1, z_2, z_3, x_1\} \). If \( z_{x_2}' = z_{x_2}'' \), then we must have \((x_2, z_{x_2}''), (z_{x_2}', x_2) \in A(D)\), and so \( D \) has a spanning trail induced by the arcs \( \{(z_1, z_2), (z_2, z_3), (z_3, x_1), (x_2, z_{x_2}''), (z_{x_2}', x_2)\} \). Therefore, we assume that \( z_{x_2}' \neq z_{x_2}'' \). If \( z_1 \in \{z_{x_2}', z_{x_2}''\} \), then we may assume by symmetry that \( \{z_1, z_3\} = \{z_{x_2}', z_{x_2}''\} \). It follows by \( c = 3 \) that \((z_1, x_2), (x_2, z_3) \in A(D)\), and so \( D \) has a spanning closed trail induced by the arcs \( \{(x_1, z_2), (z_2, z_3), (z_3, x_1), (x_2, z_{x_2}''), (z_{x_2}', x_2)\} \). If \( z_1 \notin \{z_{x_2}', z_{x_2}''\} \), then by \( c = 3 \) and as \( D \) is not supereulerian, we must have that \((x_2, z_2), (z_3, x_2) \in A(D)\). Since \( G(D) \) has a 5-cycle, there must be an arc \( e \in A(D) \) incident with two vertices in \( \{z_1, x_1, x_2\} \). By symmetry, assume that \((x_1, x_2) \in A(D)\), then \( D \) has a spanning trail induced by the arcs \( \{(x_1, x_2), (x_2, z_2), (z_2, z_3), (z_3, x_1)\} \). This completes the proof of the lemma.

\[ \Box \]

A **block** of a graph \( G \) is a maximal subgraph \( H \) of \( G \) such that \( H \) contains no cut vertices of itself. By definition, if \( B \) is a block of a graph \( G \) with at least 3 vertices, then \( B \) must be 2-connected. Also by definition, if \( D \) is strong, then every block of \( G(D) \) must either be 2-connected, or spanned by a 2-cycle. The main purpose of this subsection is to prove Theorem 3.18 below, which implies Theorem 3.1(i).

**Theorem 3.18** Let \( n > 1 \) be an integer, \( D \) be a strong digraph on \( n \) vertices with \( n = |V(D)| \), \( \alpha'(D) \leq 2 \) and \( s(G(D)) \geq 2 \), and \( G = G(D) \). Then one of the following holds.

(i) \( \alpha'(D) = 1 \) and \( D \) is strongly trail-connected.

(ii) \( \alpha'(D) = 2 \) and the following are equivalent.

(ii-1) \( D \) has a spanning trail.

(ii-2) \( D \notin \mathcal{D}(n) \).

**Proof.** Suppose first that \( \alpha'(D) = 1 \). Then \( G \) is spanned by a \( K_{1,n-1} \). As (i) holds trivially if \( n = 2 \), we assume that \( n \geq 3 \). Let \( v_0 \) be the vertex of degree \( n - 1 \) in this \( K_{1,n-1} \). If \( G \) does not have a cycle of length longer than 2, then \( v_0 \) is incident with every arc in \( A(D) \). As \( D \) is strong, every arc of \( D \) is symmetric, and \( D \) is the symmetric core of itself. It follows from Lemma 3.3(iii) that \( D \) is strongly trail-connected. Hence we assume that \( G \) contains a cycle of length at least 3. Then \( D \) has an arc that is not incident with \( v_0 \). By \( \alpha'(D) = 1 \), we must have \( n = 3 \) and so \( D \) is spanned by a directed 3-cycle. Once again we have that \( D \) is strongly trail-connected. This proves (i).

To prove (ii), we assume that \( \alpha'(D) = 2 \). By (18), every member \( D \in \mathcal{D}(n) \) does not have a spanning trail, and so (ii-1) implies (ii-2). Hence we assume that \( D \notin \mathcal{D}(n) \) to show that \( D \) has a spanning trail.
As it is routine to verify that every strong digraph with at most 3 vertices is supereulerian, we assume that $n \geq 4$.

Let $c = c(G)$ denote the length of a longest cycle of $G$. Since $D$ is strong and $\alpha'(G) = \alpha'(D) = 2$, $2 \leq c \leq 5$. If $c = 2$, then $\tilde{G}$, the simplification of $G$, must be a tree and so every pair of adjacent vertices $u, v \in V(D)$ are vertices of a 2-cycle in $D$. It follows by Lemma 3.3(i) that $D = J(D)$ is supereulerian. Thus we may assume that $3 \leq c \leq 5$. Let $B$ be a block of $G$ that contains a longest cycle of $G$.

**Claim 4** Each of the following holds.
(i) If $c = 5$, then $G = B$ with $|V(G)| = 5$.
(ii) If $c = 4$, then either $G = B$, or $B$ is spanned by a $K \cong K_{2,t}$ for some $t \geq 2$ with $v_1, v_2$ being two nonadjacent vertices of degree $t$ in $K$, such that every block $B'$ of $G$ other than $B$ is a 2-cycle in $D$ and contains exactly one vertex $v_{B'} \in V(K)$. Furthermore, if $t \geq 3$, then $v_{B'} \notin \{v_1, v_2\}$.

Suppose that $c = 5$ and let $C$ be a cycle of length 5. If $|V(B)| > 5$, then as $B$ is connected, an edge $e \in E(B) - E(C)$ together with a matching of size 2 not adjacent with $e$ forms a matching of sizes 3 in $B$, leading to a contradiction that $2 = \alpha'(G) \geq \alpha'(B) \geq 3$. Hence we must have $|V(B)| = 5$. Assume that $G$ has a block $B_1$ other than $B$. Then there must be an edge $e' \in E(B_1)$. By definition of blocks, $|V(B) \cap V(B_1)| \leq 1$. Since $C$ contains a matching $M'$ of size 2. It follows that $2 = \alpha'(G) \geq |M' \cup \{e'\}| = 3$, a contradiction. Hence we must have $G = B$.

Now we assume that $c = 4$, and so $B$ contains a $K_{2,2}$ as a subgraph. Choose a maximum value $t$ such that $B$ contains a subgraph $K$ isomorphic to a $K_{2,t}$. Let $w_1, w_2$ denote two nonadjacent vertices of degree $t$ in $K$ and let $V(K) - \{w_1, w_2\} = \{v_1, v_2, \ldots, v_t\}$. If there exists a vertex $z \in V(B) - V(K)$, then since $\kappa(B) \geq 2$, there will be two internally disjoint shortest paths from $z$ to two distinct vertices $z', z''$ in $V(K)$, implying that either $B$ has a cycle of length at least 5, or $G$ has a subgraph isomorphic to a $K_{2,t+1}$. As either case leads to a contradiction, we conclude that $B$ is spanned by $K$.

Assume that $G \neq B$. Let $B'$ be an arbitrary block of $G$ other than $B$. If $V(B') \cap V(B) = \emptyset$, then an edge in $B'$ together with a 2-matching in $B$ would lead to the contradiction $2 = \alpha'(D) \geq 3$. Hence every block $B'$ other than $B$ in $G$ must contain a vertex $v_{B'}$ such that $V(B') \cap V(K) = V(B') \cap V(B) = \{v_{B'}\}$, and every edge in $B'$ is incident with the vertex $v_{B'} \in V(K)$. Again by $\alpha'(D) = 2$, if $t \geq 3$, then we must have $v_{B'} \notin \{w_1, w_2\}$ for any block $B'$ other than $B$ in $G$. As $D$ is strong, $G$ is 2-edge-connected and so $\kappa'(B') \geq 2$. This implies that $B'$ is a 2-cycle containing $v_{B'}$. Since $D$ is strong, this 2-cycle in $B'$ is a 2-cycle in $D$. This justifies Claim 4.

By Claim 4 and Lemma 3.17, if $c = 5$, then $D$ has a spanning trail. Hence it suffices to assume that $3 \leq c \leq 4$ to prove Theorem 3.18(ii).

**Claim 5** Suppose that $c = 3$. Each of the following holds.
(i) Every block of $G$ has 2 or 3 vertices.
(ii) There are at most two blocks of order 3, and if $G$ has two blocks $B', B''$ of order 3, then $|V(B') \cap V(B'')| = 1$.
(iii) $D$ has a spanning closed trail.

Assume that $c = 3$. Let $B_1, B_2, \ldots, B_b$ be all the blocks of $G$ such that for some $b'$ with $1 \leq b' \leq b$, $|V(B_1)| \geq \ldots \geq |V(B_{b'})| \geq 3$ and $|V(B_{b'+1})| = \ldots = |V(B_b)| = 2$. For each $B \in \{B_1, \ldots, B_b\}$, as $c = 3$,
Suppose first that there exists a vertex \( x \in V(B) - V(C) \), then as \( \kappa(B) \geq 2 \), there will be two internally disjoint shortest paths from \( v \) to two distinct vertices in \( V(C) \), implying the \( B \) has a cycle of length at least 4. Hence we must have \( V(B) = V(C) \), and so Claim 5(i) follows.

Since two distinct blocks \( B', B'' \) of \( G \) must satisfy \( |V(B') \cap V(B'')| \leq 1 \), it follows that \( b' \leq \alpha'(D) = 2 \). Furthermore, assume that \( |V(B') \cap V(B'')| = 0 \), then as \( G \) is connected, there must be an additional block \( B''' \) of \( G \). It follows by \( |V(B')| = |V(B'')| = 3 \) and \( |V(B'''|) = 2 \) that \( G \) has a matching of size 3, contrary to \( \alpha'(D) = 2 \). This justifies Claim 5(ii).

Since \( D \) is strong, every block \( B \) of \( G \) induces a strong subdigraph \( D[V(B)] \) of \( D \). It follows by \( |V(B)| \leq 3 \) that every \( D[V(B)] \) is supereulerian. Thus \( D \) has a spanning closed trail. This completes the proof of Claim 5.

By Claims 4 and 5 and by Lemma 3.17, we may assume that \( c = 4 \). By Claim 4(ii), for some integer \( t \geq 2 \), \( G(D) \) has a unique block \( B \) spanned by a \( K_{2,t} \). If \( t = 2 \), then \( B \) is a 4-cycle. By Claim 4(ii) and Corollary 3.16, \( D \) is supereulerian, and so \( D \) has a spanning trail.

Hence we assume that \( t \geq 3 \). Let \( w_1, w_2 \) denote the two vertices of degree \( t \) in this \( K_{2,t} \) such that every block of \( G(D) \) other than \( B \) is a 2-cycle of \( D \) containing \( w_1 \) or \( w_2 \). By Example 3.13 (and using the notation in Example 3.13), \( B = D(t_1, t_1', t_1'', t_2', t_2, t_3', t_3) \) for some non negative integers \( t_1, t_1', t_1'', t_2', t_2, t_3', t_3 \) satisfying \( |V(B)| = 2 + t_1 + t_1'' + t_2 + t_2' + t_3' + t_3 \). As \( D \notin \mathcal{D}(n) \), we conclude that either (14) or (15) must hold. By Example 3.13(ii), \( D \) has a spanning trail. This completes the proof for Theorem 3.18(ii).

### 3.4 Supereulerian digraphs and strongly trail-connected digraphs

The main result of this subsection is to prove Theorem 3.1(iii) and (iv), restated in Theorem 3.19 below. Recall that \( D_0 \) denotes the vertex disjoint union of three complete digraphs of order 3.

**Theorem 3.19** Let \( D \) be a strong digraph on \( n \) vertices with \( \alpha'(D) \geq 3 \), and \( n \geq 2\alpha'(D) + 3 \), and let \( J = J(D) \) be a symmetric core of \( D \). Each of the following holds.

(i) If \( \lambda(D) \geq \alpha'(D) - 1 \), then \( D \) is supereulerian.

(ii) If \( \lambda(D) \geq \alpha'(D) \geq 4 \), then \( J \) is a spanning subdigraph of \( D \).

**Proof.** Let \( k = \alpha'(D) \geq 3 \) and \( n = |V(D)| \geq 2k + 3 \). By Corollary 3.8, Theorem 3.19(ii) holds. It suffices to prove Theorem 3.19(i). As \( \lambda(D) \geq k - 1 \geq 2 \), \( D \neq D_0 \) and for any vertex \( v \in V(D) \), \( d_D(v) \geq 2k - 2 \). Suppose first that there exists a vertex \( x_1 \in X \) such that \( d_D(x_1) \geq 2k - 1 \). If \( k_1(x_1) > 0 \), then by Lemma 3.6(iv), \( D \) is supereulerian; if \( k_1(x_1) = 0 \), then by Lemma 3.9(iv) and as \( \lambda(D) \geq 2 \), \( D \) is supereulerian. Therefore, we assume that for any vertex \( x \in X \), \( d_D(x) = 2k - 2 \). If there exists a vertex \( x_1 \in X \) with \( k_1(x_1) > 0 \), then by Lemma 3.12(ii), \( D \) is supereulerian. Now assume that for any vertex \( x \in X \), \( k_1(x) = 0 \). By Lemmas 3.10(iii) and 3.11(iii), \( D \) must also be supereulerian. This completes the proof of Theorem 3.19.
3.5 Spanning trails in digraphs

The purpose of this subsection is to prove Theorem 3.1(ii). Throughout this subsection, \(D\) denotes a strong digraph on \(n\) vertices with \(n = |V(D)| \geq 6\) and \(\alpha'(D) = k \geq 3\).

In chapter 2, we presented Example 2.11 which showed that there exists a family of digraphs \(D(k_1,k_2,\ell)\) such that for every digraph in \(D \in D(k_1,k_2,\ell)\) is a not supereulerian, also Hong et al. [30] showed that every digraph in \(D_0(k_1,k_2,2)\) is a not supereulerian.

Let \(k \geq 3\) be an integer. It is routine to verify the following.

**Observation 3.20** Every digraph \(D \in D_0(k-1,k-1,2)\) with \(\lambda(D) \geq k - 1\) has a spanning trail.

By using Example 2.11 for the structure of \(D\), we let \(D_1 \cong D_2 \cong K_k^*\) and \(U = \{u_1,u_2\}\) with an arc \((v',v'') \in (V(D_1),V(D_2))_D\), one can start with a vertex \(w'' \in V(D_2) - \{v''\}\), traverses every vertices in \(D_2\) and then passes \(u_2\); then from \(u_2\) to a vertex \(w' \in V(D_1) - \{v'\}\) and traverses every vertex in \(V(D_1)\) with the last vertex in \(v'\); and finally completes the trail with the arcs \((v',v''),(v'',u_1)\). Thus \(D\) has a spanning trail.

To prove Theorem 3.1(ii), we used Example 2.11, Theorem 2.40 and Observation 3.20.

**Proof of Theorem 3.1(ii).** Assume that \(n = |V(D)| \geq 12\), \(\alpha'(D) = k \geq 3\) and \(\lambda(D) \geq k - 1 \geq 2\). By Theorem 3.1(iii), if \(n = |V(D)| \geq 2k + 3\), then \(D\) is supereulerian and so has a spanning trail. Hence we assume that \(2k \leq n \leq 2k + 2\). If \(n \in \{2k,2k+1\}\), then by Theorem 2.40, \(D\) is supereulerian. Therefore we assume that \(n = 2k + 2\), and so by \(n \geq 12\), \(\min\{\delta^+(D),\delta^-(D)\} \geq \lambda(D) \geq k - 1 \geq \frac{n-4}{2} \geq 4\) and \(\delta^+(D) + \delta^-(D) \geq n - 4\). By Theorem 2.40, either \(D\) is supereulerian or \(D \in D_0(k-1,k-1,2)\). By Observation 3.20, \(D\) has a spanning trail. This completes the proof of Theorem 3.1(ii).
Chapter 4

4 Supereulerian Digraph Strong Product

In this chapter, we motivate an open problem Problem 6 of [26], which was raised to find natural conditions for the product of graphs to be hamiltonian. Alsatami et al. [6] showed sufficient conditions on digraphs $D_1$ and $D_2$ and proved Theorem 2.64, in chapter 2, of Cartesian product of $D_1$ and $D_2$ is supereulerian. This motivates us to present sufficient conditions on digraphs $D_1$ and $D_2$ and prove the Strong product of $D_1$ and $D_2$ is supereulerian, which is following main result of this chapter.

**Theorem 4.1** Let $D_1$ and $D_2$ be strong digraphs. If $f(D_2) \leq |V(D_1)|$ and if for some cycle factor $F$ of $D_1$, $D_1/F$ is hamiltonian, then the strong product $D_1 \boxtimes D_2$ is supereulerian.

4.1 Lemmas

In this section, we develop some lemmas which will be used in our arguments. The proof of Theorem 4.1 will be given in the last section.

Let $k \geq 0$ be an integer. We use $\mathbb{Z}_k = \{1, 2, \ldots, k\}$ to denote the cyclic group of order $k$ and with the additive binary operation $+_{\mathbb{Z}_k}$ and with $k$ being the additive identity in $\mathbb{Z}_k$. Let $H$ and $H'$ denote two digraphs. As we are to discuss product for digraphs $D_1$ and $D_2$ with $u \in V(D_1)$ and $v \in V(D_2)$, we save the notation $(u, v)$ for a vertex in the product of $D_1$ and $D_2$. Define $H \cup H'$ to be the digraph with $V(H \cup H') = V(H) \cup V(H')$ and $A(H \cup H') = A(H) \cup A(H')$.

Let $T = v_1v_2 \cdots v_k$ denote a trail. We use $T[v_1, v_k]$ to emphasize that $T$ is oriented from $v_1$ to $v_k$. For any $1 \leq i \leq j \leq k$, we use $T[v_i, v_j] = v_i v_{i+1} \cdots v_{j-1} v_j$ to denote the sub-trail of $T$. Likewise, if $Q = u_1u_2 \cdots u_k u_1$ is a closed trail, then for any $i, j$ with $1 \leq i < j \leq k$, $Q[u_i, u_j]$ denotes the subtrail $u_i u_{i+1} \cdots u_{j-1} u_j$. If $T' = w_1w_2 \cdots w_k$ is a trail with $v_k = v_1$ and $V(T) \cap V(T') = \{v_k\}$, then we use $TT'$ or $T[v_1, v_k]T'[v_k, w_k']$ to denote the trail $v_1v_2 \cdots v_kw_2 \cdots w_k'$. If $V(T) \cap V(T') = \emptyset$ and there is a path $z_1z_2 \cdots z_t$ with $z_2, \ldots, z_{t-1} \notin V(T) \cup V(T')$ and with $z_1 = v_k$ and $z_t = w_1$, then we use $Tz_1z_2 \cdots z_t T'$ to denote the trail $v_1v_2 \cdots v_kw_2 \cdots w_k'$. In particular, if $T$ is a $(v,w)$-trail of a digraph $D$ and $uv, wz \in A(D) - A(T)$, then we use $uvTwz$ to denote the $(u,z)$-trail $D[A(T) \cup \{uv, w, z\}]$. The subdigraphs $uvT$ and $Twz$ are similarly defined.

**Lemma 4.2** Let $J_1, J_2, \ldots, J_k$ be vertex disjoint strong subdigraphs of a digraph $D$, and $J = \bigcup_{i=1}^k J_i$ is the disjoint union of these subdigraphs. Let $v_1, v_2, \ldots, v_k$ be vertices in $V(D/J)$ such that for each $i \in [k]$, $J_i$ is the preimage of $v_i$. Suppose that $C' = v_1v_2 \cdots v_i$ be a cycle of $D/J_i$. Each of the following holds.

(i) $D$ has a cycle $C$ with $A(C') \subseteq A(C)$ such that for each $i \in [k]$, $V(C) \cap V(J_i) \neq \emptyset$. (Such a cycle $C$ is called a lift of the cycle $C'$.)

(ii) If for each $i \in \mathbb{Z}_k$, $e_i = v_i''v_{i+1}' \in A(C')$ is an arc in $D$ with $v_i'' \in V(J_i)$ and $v_{i+1}' \in V(J_{i+1})$, then $C[v_i', v_i'']$ is a path in $J_i$.

**Proof.** As (i) implies (ii), it suffices to prove (i). Let $C' = v_1v_2 \cdots v_kv_1$ be a cycle of $D/J$, and for each
cyclically connected if for every pair $x, y$ of distinct vertices of $D$ there is a sequence of cycles $C_1, C_2, \ldots, C_k$ such that $x$ is in $C_1$, $y$ is in $C_k$, and $C_i$ and $C_{i+1}$ have at least one common vertex for every $i \in [k-1]$. The following results are useful.

**Lemma 4.3** Let $D$ be a digraph.

(i) (Exercise 1.17 of [9]) A digraph $D$ is strong if and only if it is cyclically connected.

(ii) If $H_1$ and $H_2$ are strong subdigraphs of $D$ with $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is also strong.

Lemma 4.3 (ii) follows immediately from definition of strong digraphs.

**Proposition 4.4** (Alsatami, Liu and Zhang, Proposition 2.1 of [6]) Let $D$ be a weakly connected digraph. Then the following are equivalent.

(i) $D$ has a cycle vertex cover.

(ii) $D$ is strong.

(iii) $D$ is cyclically connected.

(iv) For any vertices $u, v \in V(D)$, there exists an eulerian chain joining $u$ and $v$.

**Lemma 4.5** Let $D_1$ and $D_2$ be digraphs. Each of the following holds.

(i) If $D_1$ and $D_2$ are cycles, then $D_1 \times D_2$ is a circulation.

(ii) If $H_1$ and $H_2$ are arc-disjoint subdigraphs of $D_1$, then $H_1 \times D_2$ and $H_2 \times D_2$ are arc-disjoint subdigraphs of $D_1 \times D_2$.

(iii) If each of $D_1$ and $D_2$ has a cycle factor, then $D_1 \times D_2$ has a cycle factor.

**Proof.** For (i), let $V_1$ and $V_2$ be the vertex sets of $D_1$ and $D_2$, respectively. It suffices to prove that for each $(u_i, v_j) \in V_1 \times V_2$, $d^{+}_{D_1 \times D_2}((u_i, v_j)) = d^{+}_{D_1 \times D_2}((u_i, v_j))$. Let $(u_i, v_j) \in V_1 \times V_2$. Since $D_1$ and $D_2$ are cycles, we have $|N^+_D(u_i)| = |N^+_D(u_i)|$ and $|N^+_D(v_j)| = |N^+_D(v_j)|$. By Definition 1.15 (ii) (Direct Product
\[ D_1 \times D_2 \), we have the following, which implies (i).
\[
d_{D_1 \times D_2}((u_i, v_j)) = |N_{D_1 \times D_2}^+(u_i, v_j)| = \left| \{(u_s, v_t) \in V_1 \times V_2 : (u_s, v_t)(u_i, v_j) \in A(D_1 \times D_2)\} \right|
\]
\[
= \left| \{(u_s, v_t) \in V_1 \times V_2 : u_su_t \in A(D_1) \text{ and } v_tv_t \in A(D_2)\} \right|
\]
\[
= \sum_{u_s \in N_{D_1}^+(u_i)} \sum_{v_t \in N_{D_2}^+(v_j)} \left| \{(u_s, v_t) \in V_1 \times V_2\} \right|
\]
\[
= |N_{D_1}^+(u_i)| \cdot |N_{D_2}^+(v_j)| = |N_{D_1}^{-}\{u_i\}| \cdot |N_{D_2}^{-}\{v_j\}|
\]
\[
= \sum_{u_s \in N_{D_1}^{-}\{u_i\}} \sum_{v_t \in N_{D_2}^{-}\{v_j\}} \left| \{(u_s, v_t) \in V_1 \times V_2\} \right|
\]
\[
= \left| \{(u_s, v_t) \in V_1 \times V_2 : u_su_t \in A(D_1) \text{ and } v_tv_t \in A(D_2)\} \right|
\]
\[
= |N_{D_1 \times D_2}^-(u_s, v_t)| = \left| \{(u_s, v_t) \in V_1 \times V_2 : (u_s, v_t)(u_i, v_j) \in A(D_1 \times D_2)\} \right|
\]
\[
= d_{D_1 \times D_2}((u_i, v_j)).
\]

to prove (ii), let \( H_1 \) and \( H_2 \) be an arc-disjoint subdigraphs of \( D_1 \). If there exists an arc
\[(u_i, v_j)(u_s, v_t) \in (H_1 \times D_2) \cap (H_2 \times D_2),
\]
then by Definition 1.15, we must have \((u_i, u_s) \in H_1 \cap H_2\). Hence if \( H_1 \) and \( H_2 \) are arc-disjoint subdigraphs of \( D_1 \), then \( H_1 \times D_2 \) and \( H_2 \times D_2 \) are arc disjoint subdigraphs of \( D_1 \times D_2 \).

To prove (iii), let \( F_1 \) and \( F_2 \) be the spanning circulations of \( D_1 \) and \( D_2 \), respectively. By Definition 1.15 (ii) (Direct product \( D_1 \times D_2 \)), \( F_1 \times F_2 \) is spanning subgraph of \( D_1 \times D_2 \). By (i), \( F_1 \times F_2 \) is a circulation, and so \( F_1 \times F_2 \) is the spanning circulation of \( D_1 \times D_2 \). Thus \( F_1 \times F_2 \) is a cycle factor of \( D_1 \times D_2 \).

\[\text{Lemma 4.6} \] Let \( D_1, D_2 \) be digraphs and \( F \) be a subdigraph of \( D_1 \). Then \( A(F \square D_2) \cap A(F \times D_2) = \emptyset \).

\[\text{Proof.} \] Suppose there exists an arc \((u_i, v_j)(u_s, v_t) \in A(F \square D_2) \cap A(F \times D_2)\). By Definition 1.15 (Cartesian Product \( D_1 \square D_2 \)) (i), as \((u_i, v_j)(u_s, v_t) \in A(F \square D_2)\), we have either \( u_i = u_s \) and \( v_jv_t \in A(D_2) \), or \( u_iu_s \in A(F) \) and \( v_j = v_t \). By Definition 1.15 (ii), if \( u_i = u_s \), or if \( v_j = v_t \), then \((u_i, v_j)(u_s, v_t) \notin A(F \times D_2) \). It follows that \( A(F \square D_2) \cap A(F \times D_2) = \emptyset \).

\[\text{Theorem 4.7} \) (Hammack, Theorem 10.3.2 of [29]) Let \( m \) and \( n \) be integers with \( m \geq n \geq 2 \) and let \( C_m \) and \( C_n \) denote the cycles of order \( m \) and \( n \), respectively. Let gcd\((m, n)\) and lcm\((m, n)\) be the greatest common divisor and the least common multiplier of \( m \) and \( n \), respectively. Then the direct product \( C_m \times C_n \) is a vertex disjoint union of gcd\((m, n)\) cycles, each of which has length lcm\((m, n)\).

We can show a bit more structural properties in the direct product revealed by Theorem 4.7, which are stated in Lemma 4.8.

\[\text{Lemma 4.8} \] Let \( D_1 \) and \( D_2 \) be digraphs with vertex set \( V_1 = \{u_1, u_2, \ldots, u_{n_1}\} \) and \( V_2 = \{v_1, v_2, \ldots, v_{n_2}\} \) (notation in (2)).

(i) Suppose that \( D_1 \) and \( D_2 \) are cycles and \( v \in V(D_2) \) is an arbitrarily given vertex. Then for any cycle
For each $u$ and $D$.

(ii) Suppose that $D_1$ and $D_2$ are circulations and $v \in V(D_2)$ is an arbitrarily given vertex. Then $D_1 \times D_2$ is also a circulation. Moreover, for any eulerian subdigraph $F$ in $D_1 \times D_2$, there exists a vertex $u \in V(D_1)$ such that the vertex $(u, v) \in V(F)$.

Proof. Suppose $D_1 = u_1u_2 \cdots u_{n_1}u_1$ and $D_2 = v_1v_2 \cdots v_{n_2}v_1$ are cycles, and by symmetry, assume that $v = v_1$. Let $C$ be a cycle in $D_1 \times D_2$. Thus $C$ contains a vertex $(u_i, v_j)$. It follows by Definition 1.15 (ii) that

$$C = (u_i, v_j)(u_{i+1}, v_{j+1}) \cdots (u_{i+n_2-j}, v_{n_2})(u_{i+n_2-j+1}, v_1) \cdots$$

where the subscripts of vertices in $D_1$ are taken in $\mathbb{Z}_{n_1}$ and those of vertices in $D_2$ are taken in $\mathbb{Z}_{n_2}$. It follows that $u = u_{i+n_2-j+1}$. This proves (i). Suppose that $D_1$ and $D_2$ are circulations. As every circulation is an arc-disjoint union of cycles (notation (1)), each of $D_1$ and $D_2$ is an arc-disjoint union of cycles. By Lemma 4.5, $D_1 \times D_2$ is also a circulation. Let $F$ be an eulerian subdigraph in $D_1 \times D_2$. By (1), $F$ is also an arc-disjoint union of cycles $C_1, C_2, \cdots$. Applying Lemma 4.8 (i) to each cycle $C_i$, we conclude that (ii) holds as well.

4.2 Proofs of Theorem 4.1

Assume that $D_1$ and $D_2$ are two strong digraphs, and for some cycle factor $F$ of $D_1$, $D_1/F$ is hamiltonian with $f(D_2) \leq |V(D_1)|$. We start with some notation for the copies of factors in the Cartesian product.

Definition 4.9 Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two strong digraphs with $V_1 = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V_2 = \{v_1, v_2, \ldots, v_{n_2}\}$. For $i \in \{1, 2\}$, let $H_i$ be a subdigraph of $D_i$.

(i) For each $u \in V_1$, let $D_2^u$ be the subdigraph of $D_1 \square D_2$ induced by $V(D_2^u) = \{(u, v) : 1 \leq i \leq n_2\}$. The subdigraph $D_2^u$ is called the $u$-copy of $D_2$ in $D_1 \square D_2$.

(ii) For each $v \in V_2$, let $D_1^v$ be the subdigraph of $D_1 \square D_2$ induced by $V(D_1^v) = \{(u, v) : 1 \leq i \leq n_1\}$. The subdigraph $D_1^v$ is called the $v$-copy of $D_1$ in $D_1 \square D_2$.

(iii) More generally, for each $u \in V_1$ (or $v \in V_2$, respectively), let $H_2^u$ (or $H_1^v$, respectively) be the subdigraph of $D_2^u$ (or $D_1^v$, respectively) induced by $A(H_2^u) = \{(u, v_i)(u, v'_i) : v_i, v'_i \in A(H_2)\}$ (or $A(H_1^v) = \{(u, v_i)(u_i, v) : u_i, v_i \in A(H_1)\}$, respectively). The subdigraph $H_2^u$ is called the $u$-copy of $H_1$ in $D_1 \square D_2$ and the subdigraph $H_1^v$ is called the $v$-copy of $H_2$ in $D_1 \square D_2$.

If two digraphs $D$ and $H$ are isomorphic, then we write $D \cong H$. The following is an immediate observation from Definition 4.9 for the Cartesian product $D_1 \square D_2$ of two digraphs $D_1$ and $D_2$.

for any $v \in V(D_2)$, $D_1 \cong D_1^v$, and for any $u \in V(D_1)$, $D_2 \cong D_2^u$. (19)

Let $F$ be a cycle factor of $D_1$ such that $D_1/F$ has a Hamilton cycle. Since $F$ is a cycle factor of $D_1$, each component of $F$ is an eulerian subdigraph of $D_1$. Let

$$F_1, F_2, \ldots, F_k$$

be the components of $F$, and $J = D_1/F$. (20)

Then $V(J) = \{w_1, w_2, \ldots, w_k\}$, where for each $i \in [k]$, $w_i$ is the contraction image in $J$ of the eulerian subdigraph $F_i$ in $D_1$. Since $J$ is hamiltonian, we may by symmetry assume that $C' = w_1w_2 \cdots w_kw_1$ is a
hamilton cycle of $J$. It follows by Lemma 4.2 that

$$D_1 \text{ has a cycle } C \text{ with } A(C') \subseteq A(C). \quad (21)$$

Now we consider $D_2$. Let $f(D_2) = m \leq |V(D_1)|$ and $F'$ be a circulation of $D_2$ such that $D_2/F'$ has a cycle vertex cover $C' = \{C_1', C_2', ..., C_m'\}$. Let $F_1', F_2', ..., F_k'$ be the components of $F'$, $w_{k'+1}'$, ..., $w_k'$ be the vertices in $V(D_2) - V(F')$. We define, for each $i$ with $k' + 1 \leq i \leq t$, $F_i'$ to be the digraph with $V(F_i') = \{w_i'\}$ and $A(F_i') = \emptyset$. With these definitions, we have

$$V(D_2/F') = \{w_1', w_2', ..., w_k', w_{k'+1}', ..., w_t'\}. \quad (22)$$

By Lemma 4.2, for each $j \in [m]$, $C_j'$ in $C'$ can be lifted to a cycle $C_j$ in $D_2$. To construct a spanning eulerian subdigraph of $D_1 \Box D_2$, we start by justifying the following claims.

**Claim 6** Each of the following holds.

(i) For any $i \in [k]$, and $j \in [t]$, $F_i \times F_j$ is a circulation.

(ii) For any $i \in [k]$, and $j \in [t]$, $F_i \sqcap F_j'$ is an eulerian digraph.

(iii) For each $i \in [k]$, and each $j \in [t]$, if $v \in V(F_j')$, then $F_i^v \cup (F_i \times F_j')$ is a spanning eulerian subdigraph of $F_i \boxtimes F_j'$.

**Proof.** For each $i \in [k]$, $F_i$ is an eulerian subdigraph of $D_1$, so $F_i$ is a disjoint union of cycles. Similarly, for each $j \in [k']$, $F_j'$ is an eulerian subdigraph of $D_2$, so $F_j'$ is a disjoint union of cycles. By Lemma 4.8, $F_i \times F_j'$ is a circulation.

By assumption, for each $i \in [k]$, $F_i$ is an eulerian subdigraph of $D_1$. If $j \in [k']$, then as $F_j'$ is an eulerian subdigraph of $D_2$, it follows by Theorem 2.63 that $F_i \sqcap F_j'$ is an eulerian digraph. Now assume that $k' + 1 \leq j \leq t$. Then $V(F_j') = \{w_j'\}$, and so by (19), $F_i \sqcap F_j' = F_i^{w_j'} \cong F_i$ is eulerian. This proves (ii).

For each $i \in [k]$, each $j \in [t]$ and a fixed vertex $v \in V(F_j')$, let $J' = F_j^v \cup (F_i \times F_j')$. By (i), $F_i \times F_j'$ is a circulation. By (19), $F_i^v \cong F_i$ is an eulerian digraph. By Lemma 4.6, $A(F_i^v) \cap A(F_i \times F_j') = \emptyset$. It follows that for any vertex $z \in V(J')$,

$$d_{J'}^-(z) = d_{F_j^v}^+(z) + d_{F_i \times F_j'}^-(z) = d_{F_j^v}^+(z) + d_{F_i \times F_j'}^-(z) = d_{J'}^-(z)$$

and so $J'$ is a circulation. Without loss of generality, we denote $V(F_i) = \{u_{i_1}, u_{i_2}, ..., u_{i_t}\}$ and $V(F_j') = \{v_{j_1}, v_{j_2}, ..., v_{j_{k'}}\}$ with $v = v_{j_1}$. To prove that $J'$ is connected, let $(u_{i_1}, v_{j_1}) \in V(J')$ and let $J_1$ be the connected component of $J'$ that contains $(u_{i_1}, v_{j_1})$. If $J'$ is not connected, then by symmetry, we may assume that there exists a vertex $(u_{j_2}, v_{j_2}) \in V(J') - V(J_1)$. As $F_i \times F_j'$ is a circulation, there must be an eulerian subdigraph $F$ of $F_i \times F_j'$ with $(u_{i_2}, v_{j_2}) \in V(F)$. By Lemma 4.8(ii), there exist a vertex $u' \in V(D_1)$ such that $(u', v_{j_1}) \in V(F)$. Thus by Definition 4.9(ii), $V(F) \cap V(F_j') = \emptyset$. By (19) and (20), $F_i^v \cong F_i$ is connected, and so both $(u_{i_1}, v_{j_1})$ and $(u', v_{j_1})$ must be in the same component of $J'$. This implies that $(u', v_{j_1}) \in V(J_1)$. Since $(u_{i_2}, v_{j_2})$ and $(u', v_{j_1})$ are in the same component of $J'$, it follows that $(u_{i_2}, v_{j_2}) \in V(J_1)$ also, contrary to the assumption that $(u_{i_2}, v_{j_2}) \in V(J') - V(J_1)$. Hence $J'$ must be connected, and so $F_i^v \cup (F_i \times F_j')$ is a spanning eulerian subdigraph $F_i \boxtimes F_j'$.

**Claim 7** Let $C'$ be a Hamilton cycle of $J$ and $C$ be a lift of $C'$ in $D_1$ as warranted by (21). For each
v ∈ V(D_2), let C^w denote the v-copy of C in D_1 □ D_2. For each j ∈ [t], if v, v' ∈ V(F'_j) are two distinct vertices, then

\[ H_{v,v';j} := \bigcup_{i=1}^{k} (F''_i ∪ (F_i × F'_j)) \cup C^w \]

is a spanning eulerian subdigraph D_1 □ F'_j.

**Proof.** By Lemma 4.2, for any i ∈ [k], V(C^w) ∩ V(F''_i) ≠ ∅. By Claim 6 (iii), for any i ∈ [k] and for any j ∈ [t], F''_i ∪ (F_i × F'_j) is a spanning eulerian subdigraph F_i □ F'_j, and so F''_i ∪ (F_i × F'_j) is a strong subdigraph of D_1 □ F'_j. Since for any i ∈ [k], V(C^w) ∩ V(F''_i) ≠ ∅, we may assume that for some vertex u ∈ V(F_i), (u, v) ∈ V(C^w) ∩ V(F''_i). As v ∈ V(F'_j), we have (u, v) ∈ V(C^w) ∩ V(F''_i ∪ (F_i × F'_j)), and so F''_i ∪ (F_i × F'_j) ∪ C^w is connected. Since v ≠ v', A(C^w) ∩ A(F''_i ∪ (F_i × F'_j)) = ∅, we conclude from the facts that C^w and F_i × F'_j are circulations (see Claim 6(i)) that F''_i ∪ (F_i × F'_j) ∪ C^w is eulerian. As i ∈ [k] is arbitrarily, we conclude that H_{v,v';j} = \bigcup_{i=1}^{k} (F''_i ∪ (F_i × F'_j)) ∪ C^w is an eulerian subdigraph with vertex set V(H_{v,v';j}) = \bigcup_{i=1}^{k} (F_i × F'_j) = V(D_1 □ F'_j). This proves Claim 7.

**Claim 8** Let u ∈ V(D_1) be an arbitrary vertex, F' be a circulation of D_2 such that D_2/F' has a cycle vertex cover C' = {C'_1, C'_2, ..., C'_m} with m = f(D_2) ≤ |V(D_1)|. Each of the following holds.

(i) F''_u is a circulation of D_2.

(ii) For any j ∈ [m], C'_j is a cycle of D_2/F''_u, and \{C'^u_1, C'^u_2, ..., C'^u_m\} is a cycle vertex cover of D_2/F''_u.

(iii) Let u ∈ V(D_1) be a vertex, h ∈ [m] be arbitrarily given. For any vertex w_j' ∈ V(C'_h), let v(j), v'(j) be two distinct vertices in V(F'_j), and C'_h be a lift of C'_h in D_2. Then

\[ H''_h = \bigcup_{w_j' ∈ V(C'_h)} H_{v(j),v'(j);j} \cup C''_h \]

is an eulerian digraph with V(H''_h) = \bigcup_{v_j' ∈ V(C'_h)} V(D''_1).

**Proof.** Each of (i) and (ii) follows from (19) and the definition of C'. It remains to prove (iii). By Lemma 4.2, C'_h can be lifted to a cycle C'_h in D_2. For any w'_j ∈ V(C'_h), pick two distinct vertices v, v' ∈ V(F'_j). By Claim 7, H_{v,v';j} defined in Claim 7 is a spanning eulerian subdigraph D_1 □ F'_j. By Lemma 4.6, C''_h = D_1[{u}] □ C'_h is arc-disjoint from each H_{v,v';j}, and so the facts that C''_h is a directed cycle and H_{v,v';j} is eulerian, it follows that H''_h is a circulation. By Definition 4.9 (iii) and by Lemma 4.6, a vertex w'_j ∈ V(C''_h) if and only if V(C''_h) ∩ V(F''_u) ≠ ∅. This is equivalent to saying that a vertex w'_j ∈ V(C'_h) if and only if for some vertex v'' ∈ V(F'_j), (u, v'') ∈ V(C'_h). Since C''_h is a cycle, and since, for each w'_j ∈ V(C''_h), there exists some vertex v'' ∈ V(F'_j) with (u, v'') ∈ V(C''_h), we observe that V(H_{v,v';j} ∩ V(C''_h)) contains a vertex (u, v''), it follows that H''_h must be connected. Hence H''_h is a connected circulation, and so it must be eulerian. To complete the justification of Claim 8 (iii), we note that by definition,

V(C''_h) ⊆ \bigcup_{w'_j ∈ V(C''_h)} V(D_1 □ F'_j).

This, together with Claim 7, implies

V(H''_h) = \bigcup_{w'_j ∈ V(C''_h)} V(H_{v(j),v'(j);j}) \cup V(C''_h) = \bigcup_{w'_j ∈ V(C''_h)} V(D_1 □ F'_j) = \bigcup_{v_j' ∈ V(C'_h)} V(D''_1).

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This completes the proof of Claim 8. □

Recall that $V(D_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ with $n_1 \geq m = f(D_2)$. We will complete the proof of Theorem 3.1 by proving that

$$H = \bigcup_{h=1}^{m} H_{n_h}$$

is a spanning eulerian subdigraph of $D_1 \boxdot D_2$. By Claim 8 (iii), we conclude that $V(H) = \bigcup_{j=1}^{t} V(D_1 \boxdot F'_j) = V(D_1 \boxdot D_2)$. As $u_1, \ldots, u_m$ are mutually distinct, and as $F'_1, F'_2, \ldots, F'_t$ are mutually vertex disjoint, we conclude that the $H_{n_h}$’s are mutually arc-disjoint. By Claim 8 (iii), each $H_{n_h}$ is eulerian, and so $H$ is a circulation. It remains to show that $H$ is connected. By Claim 8 (iii), $H$ has a component $H'$ that contains $H_{n_h}$. If $H = H'$, then done. Assume that $V(H) - V(H') \neq \emptyset$. Since $H'$ is a component, if some $H_{n_h}$ contains a vertex in $H'$, then $H'$ contains $H_{n_h}$ as a subdigraph. Thus every $H_{n_h}$ is either contained in $H'$ or totally disjoint from $H'$. Let $W = \{w'_j \in V(D_2/F') : H_{n_h}^{j'}$ is contained in $H'\}$. Then as $H \neq H'$, $V(D_2/F') - W \neq \emptyset$. Since $C'$ is a cycle vertex cover of $D_2/F'$, it follows by Definition 1.12 (ii) that there must be a cycle $C'_j \in C'$ such that $C'_j$ contains a vertex $w' \in W$ and a vertex $w'' \in V(D_2/F') - W$. Since $w' \in W$, $H_{n_h}^{j'}$ is contained in $H'$. Since $w', w'' \in V(C'_j)$, it follows that $w'' \in W$, contrary to the fact that $w'' \in V(D_2/F') - W$. This contradiction indicates that we must have $H = H'$, and so $H$ is a spanning eulerian subdigraph of $D_1 \boxdot D_2$. □
References


