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Finite-temperature Feynman propagator in operator form

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.

THE FINITE-TEMPERATURE FEYNMAN PROPAGATOR IN OPERATOR FORM

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In momentum space the Feynman propagator $D_F(k)$ at non-zero temperature is defined by a simple dispersion relation with the familiar property of being an even function of k^0 and analytic for $\mathrm{Re}(k^0)^2 > 0$. The coordinate space form of the propagator $D_F(x)$ is expressed directly in terms of matrix elements of the field operator and requires a new type of operator ordering.

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I. INTRODUCTION

In zero-temperature field theory the Feynman propagator is the vacuum expectation value of the time-ordered product of two field operators. Therefore it is rather curious that at non-zero temperature the Feynman propagator and the time-ordered propagator propagator are different. At $T \neq 0$ the time-ordered propagator for a real scalar field ϕ is

$$iD_{11}(x) = \sum_N \langle N| T\big(\phi(x)\phi(0)\big)|N\rangle \, \frac{e^{-\beta E_N}}{Z} \tag{1}$$

where $|N\rangle$ are eigenstates of the total Hamiltonian and β is the inverse temperature. In momentum space it has the dispersion representation [1,2]

$$D_{11}(k) = \int_{-\infty}^{\infty} d\omega \Big(\frac{1+f(\omega)}{k^0-\omega+i\eta} - \frac{f(\omega)}{k^0-\omega-i\eta} \Big) \, \rho(\omega,\vec{k}) \tag{2}$$

where $f(\omega) = 1/[e^{\beta\omega}-1]$ and the spectral function is

$$\rho(k) = \frac{1}{2\pi} \sum_N \int d^4x \; e^{ik\cdot x} \langle N| \big[\phi(x),\phi(0)\big]|N\rangle \, \frac{e^{-\beta E_N}}{Z} \tag{3}$$

Because $\rho(k)$ is real, it is easy to separate the real and imaginary parts of (2):

$$\begin{aligned} \mathrm{Re}\, D_{11}(k) &= \mathcal{P}\int_{-\infty}^{\infty} d\omega \; \frac{\rho(\omega,\vec{k})}{k^0-\omega} \\ \mathrm{Im}\, D_{11}(k) &= -\pi \coth(\frac{\beta k^0}{2}) \; \rho(k) \end{aligned} \tag{4}$$

To explain what is meant by the Feynman propagator at non-zero temperature it is helpful to recall that real-time calculations require doubling the number of degrees of freedom [2-9]. Associated with each physical field is an auxiliary field. All propagators become 2×2 matrices in the internal space. Thus D_{11} is one entry in the 2×2 matrix. In momentum space it has the form [2-9]

$$D_{ad}(k) = U_{ab} \begin{bmatrix} D_F(k) & 0 \\ 0 & -D_F^*(k) \end{bmatrix}_{bc} U_{cd} \tag{5}$$

where $D_F(k)$ is the Feynman propagator. The simplicity of this matrix structure indicates that the thermal Feynman propagator plays a central role in thermal field theory. To extract a representation for $D_F(k)$, use $U_{12} = U_{21} = [\exp(\beta|k^0|) - 1]^{-1/2}$ and $U_{11} = U_{22} = \exp(\beta|k^0|/2)\, U_{12}$ to obtain

$$D_{11}(k) = \frac{D_F(k)\exp(\beta|k^0|) - D_F^*(k)}{\exp(\beta|k^0|) - 1} \tag{6}$$

When the real and imaginary parts of this relation are compared to (4) the result is

$$\begin{aligned} \mathrm{Re}\, D_F(k) &= \mathcal{P}\int_{-\infty}^{\infty} d\omega\ \frac{\rho(\omega,\vec{k})}{k^0 - \omega} \\ \mathrm{Im}\, D_F(k) &= -\pi\epsilon(k^0)\ \rho(k) \end{aligned} \tag{7}$$

Consequently the thermal Feynman propagator satisfies the dispersion relation

$$D_F(k) = \int_{-\infty}^{\infty} d\omega \frac{\rho(\omega,\vec{k})}{k^0 - \omega + i\eta\epsilon(k^0)} \tag{8}$$

Using $\rho(-k^0,\vec{k}) = -\rho(k^0,\vec{k})$ this can also be written

$$D_F(k) = \int_0^{\infty} d\omega \frac{2\omega\ \rho(\omega,\vec{k})}{(k^0)^2 - \omega^2 + i\eta} \tag{9}$$

This dispersion relation is not new; it is discussed in [2] for example. It shows that the thermal Feynman propagator is an even function of k^0 that is analytic in the region $\mathrm{Re}(k^0)^2 > 0$. The dispersion relation has exactly the same appearance as at zero temperature because all dependence on the temperature is contained in the thermal spectral function. (In practice it is more complicated than at zero temperature because there are no regions of ω where the spectral function vanishes.)

Since D_F has rather simple properties and plays a central role in the matrix (5), it seems worthwhile to ask how D_F can be expressed directly in terms of matrix elements of the field operator $\phi(x)$. The answer to this question is given below in (10). The proof of (10) is that the Fourier transform produces the defining dispersion relation (8). The Appendix contains a discussion of how to extract the operator form for the Feynman proper self-energy $\Pi_F(x)$.

II. OPERATOR FORM FOR $D_F(x)$

In coordinate space the Feynman propagator is the thermal average of a certain ordered product of two field operators. The ordering is, however, more complicated than the usual time-ordering. The result is

$$\begin{aligned} iD_F(x) = \sum_N \Big[& \langle N|\phi(x)\ \theta_N(t)\ \phi(0)|N\rangle \\ & + \langle N|\phi(0)\ \theta_N(-t)\ \phi(x)|N\rangle \Big]\ \frac{e^{-\beta E_N}}{Z} \end{aligned} \tag{10}$$

The new ordering operation is defined by

$$\theta_N(t) = \theta(t)\theta(H - E_N) - \theta(-t)\theta(E_N - H) \tag{11}$$

Having the Hamiltonian operator in the argument of the theta function is a compact way of representing a projection operator. For example

$$\theta(H - E_N) = \sum_{E_A > E_N} |A\rangle\langle A| \tag{12}$$

Since the ordering depends only on the energy E_N and not on a particular state $|N\rangle$, it could be labelled $\theta_{E_N}(x^0)$, but that seems cumbersome. The time derivative of (11) is particularly simple:

$$\frac{d}{dt}\theta_N(t) = \delta(t) \tag{13}$$

Note that (10) can be written in a variety of forms by inserting a complete set of states (12) and regrouping the matrix elements in various ways. In the zero-temperature limit (10) reduces to the usual time-ordered product as it should. In this limit the only state $|N\rangle$ that contributes is the vacuum. Since $\theta(H - E_{vac}) = 1$ and $\theta(E_{vac} - H) = 0$, it follows that $\theta_N(t) \to \theta(t)$ at zero temperature.

The remainder of the discussion will prove that the Fourier transform of (10) yields the dispersion relation (8). To do this it is simplest to transform only the time-dependence of (10) without changing the space-dependence

$$D_F(k^0, \vec{x}) = \int_{-\infty}^{\infty} dt\ e^{ik^0 t}\ D_F(x) \tag{14}$$

Examine the first term $\langle N|\phi(x)\theta_N(t)\phi(0)|N\rangle$ in (10). Using the time-dependence of the Heisenberg field, $\phi(x) = \exp(iHt)\phi(\vec{x})\exp(-iHt)$, this can be written

$$\langle N|\phi(x)\theta_N(t)\phi(0)|N\rangle = \langle N|\phi(\vec{x})e^{i(E_N-H)t}\;\theta_N(t)\phi(0)|N\rangle$$

the Fourier transform is

$$\int_{-\infty}^{\infty} dt\; e^{ik^0t}\langle N|\phi(x)\theta_N(t)\phi(0)|N\rangle = i\langle N|\phi(\vec{x})\; R\; \phi(0)|N\rangle$$

where

$$R \equiv \frac{\theta(H-E_N)}{k^0+E_N-H+i\eta} + \frac{\theta(E_N-H)}{k^0+E_N-H-i\eta}$$

As before, the appearance of the Hamiltonian operator can be replaced by summing over a complete set of states. One can write R more compactly as

$$R = \int_{-\infty}^{\infty} d\omega\; \frac{\delta(\omega+E_N-H)}{k^0-\omega+i\eta\epsilon(k^0)} \tag{15}$$

In the same way the Fourier transform of the second term in (10) is

$$\int_{-\infty}^{\infty} dt\; e^{ik^0t}\langle N|\phi(0)\theta_N(-t)\phi(x)|N\rangle = -i\langle N|\phi(0)\; S\; \phi(\vec{x})|N\rangle$$

where

$$S = \int_{-\infty}^{\infty} d\omega\; \frac{\delta(\omega-E_N+H)}{k^0-\omega+i\eta\epsilon(k^0)} \tag{16}$$

The propagator (10) thus has the form

$$D_F(k^0,\vec{x}) = \int_{-\infty}^{\infty} d\omega\; \sum_N \frac{e^{-\beta E_N}}{Z}\frac{f_N(\omega,\vec{x})}{k^0-\omega+i\eta\epsilon(k^0)} \tag{17}$$

where

$$\begin{aligned} f_N(\omega,\vec{x}) =& \langle N|\phi(\vec{x})\delta(\omega+E_N-H)\phi(0)|N\rangle \\ & -\langle N|\phi(0)\delta(\omega-E_N+H)\phi(\vec{x})|N\rangle \end{aligned}$$

It is easy to see that f_N is the Fourier transform of the commutator:

$$\begin{aligned} f_N(\omega,\vec{x}) &= \int_{-\infty}^{\infty}\frac{dt}{2\pi}\; e^{i\omega t}\Big(\langle N|\phi(\vec{x})e^{i(E_N-H)t}\phi(0)|N\rangle \\ &\qquad\qquad -\langle N|\phi(0)e^{i(H-E_N)t}\phi(\vec{x})|N\rangle\Big) \\ &= \int_{-\infty}^{\infty}\frac{dt}{2\pi}\; e^{i\omega t}\langle N|\big[\phi(x),\phi(0)\big]|N\rangle \end{aligned}$$

where the time dependence of $\phi(x)$ was used in the last step. Thus the integrand of (17) contains the thermal spectral function:

$$\sum_N \frac{e^{-\beta E_N}}{Z} f_N(\omega, \vec{x}) = \int_{-\infty}^{\infty} dt e^{i\omega t} \rho(t, \vec{x}) = \rho(\omega, \vec{x}) \tag{18}$$

Consequently

$$D_F(k^0, \vec{x}) = \int_{-\infty}^{\infty} d\omega \, \frac{\rho(\omega, \vec{x})}{k^0 - \omega + i\eta\epsilon(k^0)} \tag{19}$$

The spatial Fourier transform of this is the defining relation (8) and thus proves that the operator form (10) is correct.

To use the operator form systematically one could develop the diagramatic rules for perturbation theory. This would require a Wick theorem for the θ_N product and an extension of the operator approach of Nieves [8] from the time-ordered to the Feynman case. It might also be interesting to find the operator form for the finite temperature Feynman propagator of quarks and gluons.

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APPENDIX A: THE FEYNMAN PROPER SELF-ENERGY

It is possible to deduce the operator form for the Feynman proper self-energy $\Pi_F(x)$ that generates $D_F(x)$. In momentum space the proper self-energy $\Pi_F(k)$ is defined by

$$D_F(k) = \frac{1}{k^2 - m^2 - \Pi_F(k)} \tag{A1}$$

The dispersion relation (9) guarantees that $\Pi_F(k)$ is analytic in the region $\mathrm{Re}(k^0)^2 > 0$ and that $\mathrm{Im}\Pi_F(k)$ is negative when k^0 real. Rewrite (A1) in the form

$$(-k^2 + m^2)D_F(k) = -1 - \Pi_F(k)\; D_F(k) \tag{A2}$$

The coordinate space form of (A2) is the Schwinger-Dyson equation:

$$(\Box + m^2)D_F(x) = -\delta^4(x) - \int d^4y\; \Pi_F(x-y) D_F(y) \tag{A3}$$

To deduce the operator form for $\Pi_F(x)$ we therefore need to apply $(\Box + m^2)$ to the operator representaion (10). A single time derivative of (10) gives

$$\begin{aligned} i\dot{D}_F(x) = \sum_N \Big[&\langle N|\dot{\phi}(x)\; \theta_N(t)\; \phi(0)|N\rangle \\ &+ \langle N|\phi(0)\; \theta_N(-t)\; \dot{\phi}(x)|N\rangle \Big] \; \frac{e^{-\beta E_N}}{Z} \end{aligned} \tag{A4}$$

Because of (13) the terms involving the time derivative of $\theta_N(t)$ give

$$\delta(t) \sum_N \langle N| \big[\phi(x),\; \phi(0)\big] |N\rangle \; \frac{e^{-\beta E_N}}{Z} = 0 \tag{A5}$$

which contains the commutator is at space-like separation and thus vanishes by causality. The second time derivative is

$$\begin{aligned} i\ddot{D}_F(x) = \sum_N \Big[&\delta(t) \langle N| \big[\dot{\phi}(x), \phi(0)\big] |N\rangle \\ &+ \langle N|\ddot{\phi}(x)\; \theta_N(t)\; \phi(0)|N\rangle \\ &+ \langle N|\phi(0)\; \theta_N(-t)\; \ddot{\phi}(x)|N\rangle \Big] \; \frac{e^{-\beta E_N}}{Z} \end{aligned} \tag{A6}$$

The first term of (A6) gives $-i\delta^4(x)$ because of the equal time commutation relations. The remaing two terms are given by the field equation

$$\ddot{\phi}(x) = (\nabla^2 - m^2)\phi(x) + J(x) \tag{A7}$$

where $J = \delta\mathcal{L}_I/\delta\phi$. Consequently (A6) becomes

$$\begin{aligned}(\Box + m^2)D_F(x) = -\delta^4(x) - i\sum_N \Big[&\langle N|J(x)\ \theta_N(t)\ \phi(0)|N\rangle \\ &+\langle N|\phi(0)\ \theta_N(-t)\ J(x)|N\rangle\Big]\ \frac{e^{-\beta E_N}}{Z}\end{aligned} \tag{A8}$$

This is just the form (A3). If D_F in (A3) were written as a function of $x - z$ instead of x only, then the integrand of the d^4y integration would be $\Pi_F(x-y)D_F(y-z)$. Comparison with (A8) gives

$$\begin{aligned}\int d^4y\,\Pi_F(x-y)D_F(y-z) = i\sum_N \Big[&\langle N|J(x)\ \theta_N(x^0-z^0)\ \phi(z)|N\rangle \\ &+\langle N|\phi(z)\ \theta_N(z^0-x^0)\ J(x)|N\rangle\Big]\ \frac{e^{-\beta E_N}}{Z}\end{aligned} \tag{A9}$$

This defines the operator form of the Feynman proper self-energy in direct analogy to the definition of the time-ordered proper self-energy [10].

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