2008

Mod \((2p + 1)\)-Orientations and \(K_{\{1,2p+1\}}\)-Decompositions

Hong-Jian Lai

Follow this and additional works at: https://researchrepository.wvu.edu/faculty_publications

Digital Commons Citation
Lai, Hong-Jian, "Mod \((2p + 1)\)-Orientations and \(K_{\{1,2p+1\}}\)-Decompositions" (2008). Faculty Scholarship. 231.
https://researchrepository.wvu.edu/faculty_publications/231

This Article is brought to you for free and open access by The Research Repository @ WVU. It has been accepted for inclusion in Faculty Scholarship by an authorized administrator of The Research Repository @ WVU. For more information, please contact ian.harmon@mail.wvu.edu.
MOD \((2p + 1)\)-ORIENTATIONS AND \(K_{1,2p+1}\)-DECOMPOSITIONS*

HONG-JIAN LAI

Abstract. In this paper, we establish an equivalence between the contractible graphs with respect to the mod \((2p + 1)\)-orientability and the graphs with \(K_{1,2p+1}\)-decompositions. This is applied to disprove a conjecture proposed by Barat and Thomassen that every 4-edge-connected simple planar graph \(G\) with \(|E(G)| \equiv 0 \pmod{3}\) has a claw decomposition.

Key words. nowhere zero flows, circular flows, mod \((2p+1)\)-orientations, \(K_{1,2p+1}\)-decompositions

AMS subject classification. 05C99

DOI. 10.1137/060676945

1. Introduction. Graphs in this paper are finite and loopless and may have multiple edges. See [2] for undefined notations and terminologies. In particular, \(\kappa'(G)\) denotes the edge connectivity of a graph \(G\), and if \(X\) is an edge subset or a vertex subset of a graph \(G\), then \(G[X]\) denotes the subgraph of \(G\) induced by \(X\). A connected loopless graph with 3 edges and a vertex of degree 3 is called a generalized claw. If restricted to simple graphs, a generalized claw must be isomorphic to a \(K_{1,3}\). A graph \(G\) with \(|E(G)| \equiv 0 \pmod{3}\) has a claw decomposition if \(E(G)\) can be partitioned into disjoint unions \(|X_1 \cup X_2 \cup \ldots \cup X_k|\) such that, for each \(i\) with \(1 \leq i \leq k\), \(|X_i|\) is a generalized claw. Barat and Thomassen [1] showed that the claw-decomposition problem is closely related to the nowhere zero 3-flow problem. In particular, the following conjecture is proposed.

Conjecture 1.1 (Barat and Thomassen [1]). Every 4-edge-connected simple planar graph \(G\) with \(|E(G)| \equiv 0 \pmod{3}\) has a claw decomposition.

The purpose of this note is to disprove this conjecture. In section 2, we shall introduce contractible graphs with respect to the mod \((2p+1)\)-orientability and discuss their properties and their relationship to the graphs with \(K_{1,2p+1}\)-decompositions. In section 3, we disprove the conjecture above.

2. \(M_{2p+1}^2\) and \(K_{1,2p+1}\)-decompositions. Throughout this section, \(p > 0\) denotes an integer. We shall extend the definition of claw decomposition to \(K_{1,2p+1}\)-decomposition as follows. A connected loopless graph with \(2p + 1\) edges and a vertex of degree \(2p + 1\) is called a generalized \(K_{1,2p+1}\). A graph \(G\) with \(|E(G)| \equiv 0 \pmod{2p + 1}\) has a \(K_{1,2p+1}\)-decomposition if \(E(G)\) can be partitioned into disjoint unions \(|X_1 \cup X_2 \cup \ldots \cup X_k|\) such that, for each \(i\) with \(1 \leq i \leq k\), \(|X_i|\) is a generalized \(K_{1,2p+1}\). In this case, we say that \(G\) has a \(K_{1,2p+1}\)-decomposition \(X = \{X_1, X_2, \ldots, X_k\}\).

Let \(D = D(G)\) be an orientation of an undirected graph \(G\). If an edge \(e \in E(G)\) is directed from a vertex \(u\) to a vertex \(v\), then let \(\text{tail}(e) = u\) and \(\text{head}(e) = v\). For a vertex \(v \in V(G)\), let

\[E_D(v) = \{e \in E(D) : v = \text{tail}(e)\}\]

and

\[E_D^-(v) = \{e \in E(D) : v = \text{head}(e)\}\].

*Received by the editors December 6, 2006; accepted for publication (in revised form) August 3, 2007; published electronically November 7, 2007.

†Department of Mathematics, West Virginia University, Morgantown, WV 26506 (hjlai@math.wvu.edu).

http://www.siam.org/journals/sidma/21-4/67694.html

© 2007 Society for Industrial and Applied Mathematics
We shall denote $d^+_G(v) = |E^+_G(v)|$ (the out degree of $v$) and $d^-_G(v) = |E^-_G(v)|$ (the in degree of $v$). The subscript $D$ may be omitted when $D(G)$ is understood from the context. Let $A$ be an (additive) Abelian group. If $f : E(G) \to A$ is a function, then the boundary of $f$ is a map $\partial f : V(G) \to A$ such that

$$\partial f(v) = \sum_{e \in E^+_G(v)} f(e) - \sum_{e \in E^-_G(v)} f(e), \forall v \in V(G).$$

Let $k > 0$ be an integer, and assume that $G$ has a fixed orientation $D$. A mod $k$-orientation of $G$ is a function $f : E(G) \mapsto \{1, -1\}$ such that for all $v \in V(G)$, $\partial f(v) \equiv 0 \pmod k$. The collection of all graphs admitting a mod $k$-orientation is denoted by $M_k$. Note that, by definition, $K_1 \in M_k$. Jaeger has conjectured [7] that every $4k$-edge-connected graph is in $M_{2k+1}$. This conjecture is still open.

Throughout this note, $\mathbb{Z}$ denotes the set of all integers. For integers $a_1, a_2, \ldots, a_k$ such that not all of them are zero, let $\gcd(a_1, a_2, \ldots, a_k)$ denote the greatest common divisor of $a_1, a_2, \ldots, a_k$. For an $m \in \mathbb{Z}$, $\mathbb{Z}_m$ denotes the set of integers modulo $m$, as well as the additive cyclic group on $m$ elements. For a graph $G$, a function $b : V(G) \mapsto \mathbb{Z}_m$ is a zero sum function in $\mathbb{Z}_m$ if $\sum_{v \in V(G)} b(v) \equiv 0 \pmod m$. The set of all zero sum functions in $\mathbb{Z}_m$ of $G$ is denoted by $Z(G, \mathbb{Z}_m)$. When $k = 2p+1 > 0$ is an odd number, we denote $M^0_{2p+1}$ by the collection of graphs such that $G \in M^0_{2p+1}$ if and only if for all $b \in Z(G, \mathbb{Z}_{2p+1})$, $\exists f : E(G) \mapsto \{1, -1\}$ such that for all $v \in V(G)$, $\partial f(v) \equiv b(v) \pmod {2p+1}$.

Note that if a function $f : E(G) \mapsto \{1, -1\}$ is given, then one can reverse the orientation of $e$ for each $e \in E(G)$ with $f(e) = -1$ to obtain an orientation $D'$ of $G$ such that for all $v \in V(G)$, $d^+_{D'}(v) - d^-_{D'}(v) = \partial f(v)$. Thus we have the following proposition.

**Proposition 2.1.** $G \in M^0_{2p+1}$ if and only if for all $b \in Z(G, \mathbb{Z}_{2p+1})$, $G$ has an orientation $D$ with the property that for all $v \in V(G)$, $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod{2p+1}$.

For a subgraph $H$ of $G$, define the set of vertices of attachments of $H$ in $G$ to be $A_G(H) = \{v \in V(H) : v$ is adjacent to a vertex in $G - V(H)\}$.

**Proposition 2.2.** For any integer $p \geq 1$, $M^0_{2p+1}$ is a family of connected graphs such that each of the following holds.

(C1) $K_1 \in M^0_{2p+1}$.

(C2) If $e \in E(G)$ and if $G - e \in M^0_{2p+1}$, then $G \in M^0_{2p+1}$.

(C3) If $H$ is a subgraph of $G$, and if $H, G/H \in M^0_{2p+1}$, then $G \in M^0_{2p+1}$.

**Proof.** (C1) and (C2) are straightforward, and so we verify only (C3).

Suppose that $G$ has a fixed orientation, $H$ is a subgraph of $G$, and both $H \in M^0_{2p+1}$ and $G/H \in M^0_{2p+1}$. Thus the edges in both $H$ and $G/H$ are oriented by the orientation of $G$. By (C2), we may assume that $H$ is an induced subgraph of $G$, and so $E(G)$ is the disjoint union of $E(H)$ and $E(G/H)$. Note that $H$ is connected, and so $H$ will be contracted to a vertex $v_H$ (say) in $G/H$. Let $b : V(G) \mapsto \mathbb{Z}_{2p+1}$ such that $\sum_{v \in V(G)} b(v) \equiv 0 \pmod {2p+1}$, and let $a_0 = \sum_{v \in V(H)} b(v)$. Define $b_1 : V(G/H) \mapsto A$ by setting $b_1(z) = b(z)$ if $z \neq v_H$, and $b_1(v_H) = a_0$. Then $\sum_{z \in V(G/H)} b_1(z) = \sum_{z \in V(G)} b(z) \equiv 0 \pmod {2p+1}$. Since $G/H \in M^0_{2p+1}$, there exists $f_1 : E(G/H) \mapsto \{1, -1\}$ such that $\partial f_1 = b_1$. For each $z \in V(H)$, define

$$b_2(z) = \begin{cases} b(z) + \sum_{e \in E^+_G(v_H) \cap E_G(z)} f_1(e) - \sum_{e \in E^-_G(v_H) \cap E_G(z)} f_1(e) & \text{if } z \in A_G(H), \\ b(z) & \text{otherwise.} \end{cases}$$

Then $\sum_{z \in V(H)} b_2(z) \equiv 0 \pmod {2p+1}$. Since $H \in M^0_{2p+1}$, there exists $f_2 : E(G/H) \mapsto \{1, -1\}$
\{1, -1\} such that \(\partial f_2 = b_2\). Now for each \(e \in E(G)\), define \(f(e) = f_1(e) + f_2(e)\). As \(E(G)\) is a disjoint union of \(E(H)\) and \(E(G/H)\), it is routine to verify that \(\partial f(z) \equiv b(z) \pmod{2p+1}\), and so \(G = M_{2p+1}^o\).

Catlin [3] (see also [4], [5]) called families of connected graphs satisfying (C1), (C2), and (C3) complete families. Complete families seem to be useful in applying certain reduction methods ([3], [4], [5]).

For a subgraph \(H\) of a graph \(G\), define

\[ \partial(H) = \{uv \in E(G) : u \in V(H), v \in V(G) - V(H)\}. \]

Let \(D\) be an orientation of \(G\). Let \(d_D^+(H)\) denote the number of edges in \(\partial(H)\) that are oriented in \(D\) from \(H\) to \(G - V(H)\), and \(d_D^-(H) = |\partial(H)| - d_D^+(H)\).

To demonstrate the relationship between \(M_{2p+1}^o\) and all of the graphs with \(K_{1,2p+1}\)-decompositions, we make the following definitions.

(i) \(k_{c,2p+1}\) denotes the smallest integer \(k > 0\) such that every \(k\)-edge-connected graph \(G\) is in \(M_{2p+1}^o\).

(ii) \(k^{c,2p+1}\) denotes the smallest integer \(k > 0\) such that every \(k\)-edge-connected graph \(G\) with \(|E(G)| \equiv 0 \pmod{2p+1}\) has a \(K_{1,2p+1}\)-decomposition.

The main result of this section is the following relationship.

**Theorem 2.3.** For any positive integer \(p > 0\), if one of \(k_{c,2p+1}\) and \(k^{c,2p+1}\) exists as a finite number, then \(k_{c,2p+1} = k^{c,2p+1}\).

To prove this theorem, we need to establish some lemmas. In each of the following lemmas, \(G\) is a graph and \(H\) is a subgraph of \(G\). Suppose that \(G\) has a \(K_{1,2p+1}\)-decomposition \(X = \{X_1, X_2, \ldots, X_k\}\), where each \(G[X_i]\) is a generalized \(K_{1,2p+1}\) for all \(i\). For each \(G[X_i]\), we orient the edges from the vertex \(v_i\) of degree \(2p+1\) in \(G[X_i]\) to all other vertices of \(G[X_i]\). This yields an orientation \(D = D(X)\) induced by the decomposition \(X\). For each \(i\), the vertex \(v_i\) is called the center of the oriented \(X_i\).

**Lemma 2.4.** Suppose that \(G\) has a \(K_{1,2p+1}\)-decomposition \(X = \{X_1, X_2, \ldots, X_k\}\), and let \(D = D(X)\). Then for any subgraph \(H\) of \(G\),

\[ |E(H)| + d_D^+(H) \equiv 0 \pmod{2p+1}. \]

**Proof.** Let \([H, G - V(H)]\) denote the set of edges in \(\partial(H)\) that are oriented in \(D(X)\) from \(H\) to \(G - V(H)\). Then \([H, G - V(H)] = d_D^+(H)\).

By the definition of \(D(X)\), the edge subset \(E(H) \cup [H, G - V(H)]\) is the disjoint union of the oriented \(X_i\)'s whose centers are in \(H\). It follows that \(|E(H)| + d_D^+(H) = |E(H)\cup[H, G - V(H)]| \equiv 0 \pmod{2p+1}\). \(\square\)

**Lemma 2.5.** Let \(b \in \mathbb{Z}\) be a number, and let \(d = |\partial(H)|\). Suppose that \(G\) has a \(K_{1,2p+1}\)-decomposition \(X\) and that \(H\) is a subgraph of \(G\). If \(2|E(H)| \equiv -d - b \pmod{2p+1}\), then, in the orientation \(D = D(X)\),

\[ d_D^+(H) - d_D^-(H) \equiv b \pmod{2p+1}. \]

**Proof.** Let \(d^+ = d_D^+(H)\) and \(d^- = d_D^-(H)\). Then \(d = d^+ + d^-\). By Lemma 2.4, \(|E(H)| \equiv -d^+ \pmod{2p+1}\), and so \(b \equiv -d - 2|E(H)| \equiv -d + 2d^+ \equiv (-d + d^+) + d^+ \equiv d^+ - d^- \pmod{2p+1}\). \(\square\)

The following result is well-known in number theory. For a reference, see Theorem 1.5 of [12].

**Lemma 2.6.** Let \(a_1, a_2, \ldots, a_k\) be integers, not all zero. Then \(gcd(a_1, a_2, \ldots, a_k) = 1\) if and only if there exist integers \(x_1, x_2, \ldots, x_k\) such that \(a_1x_1 + a_2x_2 + \cdots + a_kx_k = 1\).

**Lemma 2.7.** Let \(k, l, p \in \mathbb{Z}\) such that \(k > 0\), \(p > 0\), and \(0 \leq l \leq 2p\). Each of the following holds.

---

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
(i) There exists a planar graph \( H \) with \( \kappa'(H) \geq k \) and \( 2|E(H)| \equiv l \pmod{2p+1} \).

(ii) There exists a simple graph \( H \) with \( \kappa'(H) \geq k \) and \( 2|E(H)| \equiv l \pmod{2p+1} \).

(iii) If \( 2 \leq k \leq 5 \), then there exists a simple planar graph \( H \) with \( \kappa'(H) \geq k \) and \( 2|E(H)| \equiv l \pmod{2p+1} \).

Proof. (i) For any integer \( n > 0 \), let \( nK_2 \) denote the connected loopless graph with two vertices and \( n \) multiple edges. Let \( s > 0 \) be an integer such that \( s(2p+1) \geq k \).

Define the desired \( H \) as follows:

\[
H = \begin{cases} 
(2ps + s + t)K_2 & \text{if } l = 2t \text{ is even,} \\
((2p+1)(s+1)-(p-t))K_2 & \text{if } l = 2t + 1 \text{ is odd.}
\end{cases}
\]

(ii) Take an integer \( m \geq 4p + 2 + k \), and let \( H_0 = K_m - W \) for some edge set \( W \subset E(K_m) \) such that \( |W| \leq 4p+1+2(2p+1) \) and \( 2(|E(K_m)| - |W|) \equiv l \pmod{2p+1} \).

(iii) Since \( \gcd(10,18,2p+1) = 1 \), by Lemma 2.6, there are integers \( a_0, b_0, c_0 \) such that \( 10a_0 + 18b_0 + (2p+1)c_0 = 1 \). Choose \( x_0 = la_0 + l(|a_0| + 1)(2p+1) \) and \( y_0 = lb_0 + l(|b_0| + 1)(2p+1) \). Then \( x_0, y_0 \) are positive integers such that

\[
10x_0 + 18y_0 \equiv l \pmod{2p+1}
\]

holds. Let \( t = (2p+1)(x_0 + y_0 + 1) \), and let \( I_{12}(i), 1 \leq i \leq t-1 \), be a graph isomorphic to icosahedron defined below (see Figure 1). Define \( H \) to be the graph obtained from \( I_{12}(1), I_{12}(2), \ldots, I_{12}(t) \) by identifying \( z_i \) and \( y_{i+1} \), \( 1 \leq i \leq t-1 \), and by adding \( x_0 + 2y_0 \) new vertices \( u_1, u_2, \ldots, u_{x_0}, v_1, v_2, \ldots, v_{y_0}, w_1, w_2, \ldots, w_{y_0} \) with \( N(u_k) = \{x_5k-4,x_5k-3,x_5k-2,x_5k-1,x_5k\} \), \( N(v_k) = \{x_5k+8k'-7,x_5k+8k'-6,x_5k+8k'-5,x_5k+8k'-4,x_5k\} \), where \( 1 \leq k \leq x_0 \) and \( 1 \leq k' \leq y_0 \).

So \( H \) is a simple planar graph with \( \kappa(H) \geq k \) and \( 2|E(H)| = 60t + 10x_0 + 18y_0 = 60(2p+1)(x_0 + y_0 + 1) + 10x_0 + 18y_0 \equiv l \pmod{2p+1} \). \( \square \)

**Lemma 2.8.** (i) Let \( k > 0 \) be an integer. If every \( k \)-edge-connected (simple) graph \( G \) with \( |E(G)| \equiv 0 \pmod{2p+1} \) has a \( K_{1,2p+1} \)-decomposition, then every \( k \)-edge-connected (simple) graph \( L \in M_{2p+1} \).

(ii) Let \( k > 0 \) be an integer. If every \( k \)-edge-connected planar graph \( G \) with \( |E(G)| \equiv 0 \pmod{2p+1} \) has a \( K_{1,2p+1} \)-decomposition, then every \( k \)-edge-connected planar graph \( L \in M_{2p+1} \).

(iii) Let \( 2 \leq k \leq 5 \). If every \( k \)-edge-connected simple planar graph \( G \) with \( |E(G)| \equiv 0 \pmod{2p+1} \) has a \( K_{1,2p+1} \)-decomposition, then every \( k \)-edge-connected simple planar graph \( L \in M_{2p+1} \).

Proof. We shall prove (i) and assume first that every \( k \)-edge-connected (simple) graph \( G \) with \( |E(G)| \equiv 0 \pmod{2p+1} \) has a \( K_{1,2p+1} \)-decomposition. By contradiction, we assume that there exists a \( k \)-edge-connected (simple) graph \( L \) such that \( L \notin M_{2p+1} \).
Therefore, \( \exists b \in Z(L, \mathbb{Z}_{2p+1}) \) such that \( L \) does not have an orientation \( D \) satisfying \( d^+_\partial(i) - d^-_\partial(i) \equiv b(i) \) (mod 2p + 1) for all \( i \in V(L) \).

Let \( l_v \in \mathbb{Z} \), with \( 0 \leq l_v \leq 2p \) such that \( l_v \equiv -b(v) - d_L(v) \) (mod 2p + 1) for all \( v \in V(L) \). By Lemma 2.7 (ii), there exists a simple graph \( H_v \) with \( |E(H_v)| \equiv l_v \equiv -b(v) - d_L(v) \) (mod 2p + 1) such that \( H_v \) is also \( k \)-edge-connected. For each \( v \in V(L) \), replace \( v \) by \( H_v \) in such a way that the resulting graph \( G \) is also a \( k \)-edge-connected (simple) graph.

Since \( b \in Z(L, \mathbb{Z}_{2p+1}) \), \( 2|E(G)| = \sum_{i \in V(L)} 2|E(H_i)| + 2|E(L)| = -\sum_{i \in V(L)} b(i) - \sum_{i \in V(L)} d_L(i) + 2|E(L)| \equiv 0 \) (mod 2p + 1). By the fact that 2 and 2p + 1 are relatively prime, \( |E(G)| \equiv 0 \) (mod 2p + 1). By the assumption of this lemma, \( G \) has a \( K_{1,2p+1} \)-decomposition \( \mathcal{X} \). By the construction of \( G \), \( |\partial(H_v)| = d_L(v) \). Since \( 2|E(H_v)| \equiv l_v \equiv -b(v) - d_L(v) \) (mod 2p + 1), it follows by Lemma 2.5 that, in the orientation \( D = D(\mathcal{X}) \) for all \( v \in V(L) \), \( d^+_\partial(i) - d^-_\partial(i) \equiv b(i) \) (mod 2p + 1), contrary to the assumption that \( L \) is a counterexample.

The proofs for (ii) and (iii) are similar except that we shall use Lemma 2.7 (i) and (iii) instead of Lemma 2.7 (ii). Thus we omit the detailed proofs.

**Lemma 2.9.** If \( G \) has an orientation \( D \) such that for all \( v \in V(G) \), \( d^+_\partial(i) \equiv 0 \) (mod 2p + 1), then \( G \) is \( K_{1,2p+1} \)-decomposable.

**Proof:** Note that if \( D \) is an orientation of \( G \), then \( E(G) = \bigcup_{i \in V(G)} E^\partial(i) \) is a disjoint union. As for all \( v \in V(G) \), \( d^+_\partial(v) \equiv 0 \) (mod 2p + 1), each \( E^\partial(v) \) is a disjoint union of generalized \( K_{1,2p+1} \)'s, and so \( G \) is \( K_{1,2p+1} \)-decomposable.

**Lemma 2.10.** Suppose that \( G \in M_{2p+1}^c \). If \( |E(G)| \equiv 0 \) (mod 2p + 1), then \( G \) has a \( K_{1,2p+1} \)-decomposition.

**Proof:** For all \( v \in V(G) \), pick an \( x(v) \in \{0, 1, \ldots, 2p\} \) such that \( d(v) \equiv x(v) \) (mod 2p + 1). Define \( b(v) = d(v) - 2x(v) \). First, we shall show that \( b \in Z(G, \mathbb{Z}_{2p+1}) \). Since \( x(v) \equiv d(v) \) (mod 2p + 1), we have \( d(v) - 2x(v) \equiv -x(v) \equiv -d(v) \) (mod 2p + 1). Note also that \( \sum_{v \in V(G)} d(v) = 2|E(G)| \equiv 0 \) (mod 2p + 1). Thus
\[
\sum_{v \in V(G)} b(v) = \sum_{v \in V(G)} (d(v) - 2x(v)) = -\sum_{v \in V(G)} d(v) \equiv 0 \text{ (mod 2p + 1)}.
\]

Hence \( b \in Z(G, \mathbb{Z}_{2p+1}) \).

Since \( G \in M_{2p+1}^c \), there exists an orientation \( D \) of \( G \) such that, under this orientation, at each vertex \( v \in V(G) \), \( d^+(v) - d^-(v) = b(v) = d(v) - 2x(v) \). Since \( d^+(v) + d^-(v) = d(v) \), we have \( 2d^+(v) = 2d(v) - 2x(v) = 2(d(v) - x(v)) \). Since 2 and 2p + 1 are relatively prime, \( d^+(v) \equiv d(v) - x(v) \equiv 0 \) (mod 2p + 1). Therefore, by Lemma 2.9, \( G \) has a \( K_{1,2p+1} \)-decomposition.

Now we can easily prove Theorem 2.3. By Lemma 2.8, \( k_{c,2p+1} \leq k_{c,2p+1} \) and by Lemma 2.10, \( k_{c,2p+1} \geq k_{c,2p+1} \). Thus Theorem 2.3 follows.

By (ii) and (iii) of Lemma 2.8 and by Lemma 2.10, and noting that the edge connectivity of a simple planar graph cannot exceed 5 (Corollary 9.5.3 of [2]), we also have the following corollary.

**Corollary 2.11.** (i) Let \( k' \) denote the smallest positive integer such that every \( k' \)-edge-connected planar graph \( G \) with \( |E(G)| \equiv 0 \) (mod 2p + 1) has a \( K_{1,2p+1} \)-decomposition, and let \( k'' \) denote the smallest positive integer such that every \( k'' \)-edge-connected planar graph is in \( M_{2p+1}^c \). Then \( k' = k'' \).

(ii) Let \( l' \) denote the smallest positive integer such that every \( l' \)-edge-connected simple planar graph \( G \) with \( |E(G)| \equiv 0 \) (mod 2p + 1) has a \( K_{1,2p+1} \)-decomposition, and let \( l'' \) denote the smallest positive integer such that every \( l'' \)-edge-connected simple planar graph is in \( M_{2p+1}^c \). Then \( l' = l'' \).
3. Planar graphs. When $p = 1$, graphs in $M_o^p$ are also called $\mathbb{Z}_3$-connected graphs [8], [10], [11]. The following has been recently proved.

Theorem 3.1 (Theorem 3 of [9]). There exists a family of 4-edge-connected simple planar graphs that are not in $M_3^o$.

In fact, the dual version of Theorem 3.1 is proved in [9]. The equivalence between Theorem 3 of [9] and Theorem 3.1 here was pointed out without a proof in [8], and a formal proof of this equivalence can be found in [6].

Corollary 3.2. There exists a 4-edge-connected simple planar graph $G$ with $|E(G)| \equiv 0 \pmod{3}$ which does not have a claw decomposition.

Proof. Suppose, to the contrary, that Conjecture 1.1 holds. Then by (ii) of Corollary 2.11, every 4-edge-connected simple planar graph must be in $M_3^o$, which contradicts Theorem 3.1.

Corollary 3.2 disproves Conjecture 1.1. In fact, we can also directly construct an infinite family of 4-edge-connected simple planar graphs $G$ with $|E(G)| \equiv 0 \pmod{3}$ which does not have a claw decomposition. We present the construction as follows.

Let $k > 0$ be an integer. For each $i$ with $1 \leq i \leq 3k$, define $H_i$ to be the graph depicted below. See Figure 2.

A graph $G = G(k)$ can be constructed from the disjoint $H_i$'s by identifying $y_i$ and $x_{i+1}$, where $x_{3k+1} = x_1$ and where $i = 1, 2, \ldots, 3k$.

Example 3.3. For each $k > 0$, $G = G(k)$ defined in Figure 3 is a 4-regular and 4-edge-connected simple planar graph with $|E(G)| \equiv 0 \pmod{3}$, and $G$ has no claw decomposition.

Proof. Suppose $G$ has a claw decomposition $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$, and let $D = D(\mathcal{X})$. Since $G$ is 4-regular, for all $v \in V(G)$, $|E_D(v)| \in \{0, 3\}$. Note that $|V(G)| = 24k$ and $|E(G)| = 48k$. Thus $G$ has $m = 48k/3 = 16k$ edge-disjoint
claws. Let $W$ denote the set of vertices $v$ with $|E_D(v)| = 0$. Then $|W| = |V(G)| - m = 24k - 16k = 8k$. Note that no two vertices in $W$ are adjacent in $G$, and so, for each $i = 1, 2, \ldots, 3k \pmod{3k}$, $|W \cap V(H_i \cup H_{i+1} - \{y_{i+1}\})| \leq 5$. It follows that $16k = 2|W| = \sum_{i=1}^{3k} |V(H_i \cup H_{i+1} - \{y_{i+1}\}) \cap W| \leq 5 \times 3k = 15k$, a contradiction.

It is an open problem whether $k_{c,2p+1}$, or, equivalently, $k_{c,2p+1}$, exists as a finite number. We conjecture that it does. In view of Corollary 3.2 and Example 3.3, we further conjecture that $k_{c,2p+1} = 4p + 1$.

REFERENCES