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Nowhere-zero 3-flows in squares of graphs

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Abstract

It was conjectured by Tutte that every 4-edge-connected graph admits a nowhere-zero 3-flow. In this paper, we give a complete characterization of graphs whose squares admit nowhere-zero 3-flows and thus confirm Tutte's 3-flow conjecture for the family of squares of graphs.

1 Introduction

All graphs considered in this paper are simple. Let $G = (V, E)$ be a graph with vertex set V and edge set E . For any $v \in V(G)$, we use $d_G(v)$, $N_G(v)$ to denote the degree and the neighbor set of v in G , respectively. The minimal degree of a vertex of G is denoted by $\delta(G)$. We use K_m for a complete graph on m vertices, P_t for a path of length t and W_4 for a graph obtained from a 4-circuit by adding a new vertex x and edges joining x to all the vertices on the circuit. We call x the center of this W_4 and each edge with x as one end is called a center edge. Let D be an orientation of G . Then the set of all edges with tails (or heads) at a vertex v is denoted by $E^+(v)$ (or $E^-(v)$). If an edge uv is oriented from u to v under D , then we say $D(uv) = u \rightarrow v$. The square of G , denoted by G^2 , is the graph obtained from G by adding all the edges that join distance 2 vertices in G . We refer the reader to [1] for terminology not defined in this paper.

Definition 1.1 *Let D be an orientation of G and f be a function: $E(G) \mapsto Z$. Then*

(1). *The ordered pair (D, f) is called a **k -flow** of G if $-k + 1 \leq f(e) \leq k - 1$ for every edge $e \in E(G)$ and $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$ for every $v \in V(G)$.*

(2). *The ordered pair (D, f) is called a **Modular k -flow** of G if for every $v \in V(G)$, $\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e) \pmod{k}$.*

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The **support** of a k -flow (Modular k -flow) (D, f) of G is the set of edges of G with $f(e) \neq 0$ ($f(e) \not\equiv 0 \pmod{k}$), and is denoted by $\text{supp}(f)$. A k -flow (D, f) (Modular k -flow) of G is **nowhere-zero** if $\text{supp}(f) = E(G)$.

For convenience, a nowhere-zero k -flow is abbreviated as a k -**NZF**. The concept of integer-flow was introduced by Tutte ([7, 8] also see [9, 4]) as a refinement and generalization of the face-coloring and edge-3-coloring problems. One of the most well known open problems in this subject is the following conjecture due to Tutte:

Conjecture 1.2 (Tutte, unsolved problem 48 in [1]) *Every 4-edge-connected graph admits a 3-NZF.*

Squares of graphs admitting 3-NZF's are to be characterized in this paper. The following families of graphs are the exceptions in the main theorem.

Definition 1.3 $\mathcal{T}_{1,3} = \{T \mid T \text{ is a tree and } d_T(v) = 1 \text{ or } 3 \text{ for every } v \in V(T)\}$

Definition 1.4 $\bar{\mathcal{T}}_{1,3} = \{T \mid T \in \mathcal{T}_{1,3} \text{ or } T \text{ is a 4-circuit or } T \text{ can be obtained from some } T' \in \mathcal{T}_{1,3} \text{ by adding some edges each of which joins a pair of distance 2 leaves of } T'\}$

The following is the main result of this paper.

Theorem 1.5 *Let G be a connected simple graph. Then G^2 admits a 3-NZF if and only if $G \notin \bar{\mathcal{T}}_{1,3}$.*

An immediate corollary of Theorem 1.5 is the following partial result to Tutte's 3-flow conjecture (Conjecture 1.2).

Corollary 1.6 *Let G be a graph. If $\delta(G^2) \geq 4$ then G^2 admits a 3-NZF.*

This research is motivated by Conjecture 1.2 and the following open problem:

Conjecture 1.7 (Zhang [11]) *If every edge of a 4-edge-connected graph G is contained in a circuit of length at most 3 or 4, then G admits a 3-NZF.*

Theorem 1.5 and the following early results are partial results of the open problem above.

Theorem 1.8 (Catlin [2]) *If every edge of a graph G is contained in a circuit of length at most 4, then G admits a 4-NZF.*

Theorem 1.9 (Lai [5]) *Every 2-edge-connected, locally 3-edge-connected graph admits a 3-NZF.*

Theorem 1.10 (Imrich and Skrekovski [3]) *Let G and H be two graphs. Then $G \times H$ admits a 3-NZF if both G and H are bipartite.*

2 Splitting operation, flow extension and lemmas

Definition 2.1 (A special splitting operation) Let G be a graph and $e = xy \in E(G)$. The graph G_{*e} is obtained from G by deleting the edge e and adding two new vertices x' and y' and adding two new edges, e_x and e_y , joining x and y' , y and x' , respectively.

Definition 2.2 Let G be a graph, let (D, f) be a 3-flow of G and let $F \subseteq E(G) \setminus \text{supp}(f)$. A 3-flow (D', f') of G is called an (F, f) -changer if $F \cup \text{supp}(f) \subseteq \text{supp}(f')$.

Lemma 2.3 ([7]) A graph G admits a k -flow (D, f_1) if and only if G admits a Modular k -flow (D, f_2) such that $f_1(e) \equiv f_2(e) \pmod{k}$ for each $e \in E(G)$.

An orientation of a graph G is called a modular 3-orientation if $|E^+(v)| \equiv |E^-(v)| \pmod{3}$, for every $v \in V(G)$. The following result appears in [4, 6, 9], but by Lemma 2.3, we can attribute it to Tutte.

Lemma 2.4 ([7]) Let G be a graph. Then G admits a 3-NZF if and only if G has a modular 3-orientation.

A **partial 3-orientation** D of G is an orientation of some edges of G satisfying $|E^+(v)| \equiv |E^-(v)| \pmod{3}$, for any $v \in V(G)$. The support of D is the set of edges oriented under D and is denoted by $\text{supp}(D)$. Clearly the partial orientation obtained by reversing every oriented edge of a partial 3-orientation is also a partial 3-orientation.

Let D be a partial 3-orientation of G and let $C = v_0v_1 \cdots v_{k-1}v_0$ be a circuit of G . A **circuit-operation** along C is defined as following: For $0 \leq i \leq k-1$, if $D(v_iv_{i+1}) = v_i \rightarrow v_{i+1} \pmod{k}$, then reverse the direction of this edge; if $(v_iv_{i+1}) \pmod{k}$ is not oriented under D , then orient it as $v_i \rightarrow v_{i+1}$; if $D(v_iv_{i+1}) = v_{i+1} \rightarrow v_i \pmod{k}$ then v_iv_{i+1} loses its orientation.

Lemma 2.5 Let G be a graph, (D, f) be a 3-flow of G and H be a subgraph of G

(1). If $H \cong W_4$ and $e \in E(H) \setminus \text{supp}(f)$ is a center edge, then an $(\{e\}, f)$ -changer exists.

(2). If H is a circuit of length 3 with $E(H) \cap \text{supp}(f) = \{e\}$, then an $(E(H) \setminus \{e\}, f)$ -changer exists.

Proof. (1). Since $H \cong W_4$, let x be the center of H and let $u_1u_2u_3u_4u_1$ be the 4-circuit $H \setminus x$. Since G has a 3-flow (D, f) , then G has a partial 3-orientation D^* with $\text{supp}(D^*) = \text{supp}(f)$. We need only to find a partial 3-orientation D' such that $\text{supp}(D^*) \cup \{e\} \subseteq \text{supp}(D')$. Since e is a center edge, without loss of generality, assume that $e = xu_1$.

First we assume $E(H) \setminus \{e\} \subseteq \text{supp}(D^*)$. Without loss of generality, assume $D^*(u_1u_2) = u_1 \rightarrow u_2$. Then $D^*(u_2x) = x \rightarrow u_2$. Otherwise, we do a circuit-operation along $u_1u_2xu_1$ and then get a needed partial 3-orientation D' of G . For the same reason, u_4 must be the tail (or head) of both u_1u_4 and xu_4 . By symmetry, we consider the following two cases.

Case 1. $D^*(u_1u_4) = u_1 \rightarrow u_4$ and $D^*(xu_4) = x \rightarrow u_4$.

We may assume that u_3 is the tail (or head) of all edges incident with it in H . Otherwise, there exists a directed 2-path xu_3u_i (or u_iu_3x) for some $i \in \{2, 4\}$. Then we do circuit-operations along xu_3u_ix (or $u_iu_3xu_i$) and along $u_1u_ixu_1$. Therefore, we get a needed partial 3-orientation of D' of G .

If all edges in H have u_3 as a tail, then we do circuit-operations along xu_1u_4x , along $u_4xu_3u_4$, along xu_3u_2x and along $u_2xu_1u_2$; If all edges in H have u_3 as a head, then we do circuit-operations along $u_1u_2u_3xu_1$ and along $u_3xu_4u_3$. In both cases, we get a needed partial 3-orientation D' of G .

Case 2. $D^*(u_1u_4) = u_4 \rightarrow u_1$ and $D^*(xu_4) = u_4 \rightarrow x$.

Similar to Case 1, we may assume u_3 be the tail (or head) of all edges incident with it in H . If all edges in H have u_3 as a tail, then we do circuit-operations along xu_1u_4x , along $u_3u_4u_1u_2u_3$ and along $u_3xu_2u_3$; If all edges in H have u_3 as a head, then we do circuit-operations along $u_1xu_2u_1$, along $u_4u_1u_2u_3u_4$ and along $u_4xu_3u_4$. In both cases, we get a needed partial 3-orientation D' of G .

If $\text{supp}(D^*)$ misses some other edges of $E(H)$, say $e^* = ab \in E(H) \setminus \text{supp}(D^*)$, then we define $D^*(ab) = a \rightarrow b$ or $b \rightarrow a$, by the proof of Case 1 and Case 2, we can find a needed D' of G .

(2). it is trivial. ■

Lemma 2.6 *For each $G \in \bar{T}_{1,3}$ and each $e_0 \in E(G)$, the graph G^2 admits a 3-flow (D, f) such that $\text{supp}(f) = E(G^2) \setminus \{e_0\}$*

Proof. Induction on $|E(G)|$. It is obviously true for graphs G with $G^2 = K_4$ (including $G = C_4$, the circuit of length 4). So, assume that $|V(G)| \geq 5$ and let D be any fixed orientation of G^2 .

Let $e = xy$ with $d_G(x) = d_G(y) = 3$. Then G_{*e} consists of two components, say G_1 and G_2 . Clearly, $G_1, G_2 \in \bar{T}_{1,3}$. Without loss of generality, let $e_0 \in E(G_1)$. By induction, G_1^2 admits a 3-flow (D, f_1) such that $\text{supp}(f_1) = E(G_1^2) \setminus \{e_0\}$ and G_2^2 admits a 3-flow (D, f_2) that $\text{supp}(f_2) = E(G_2^2) \setminus \{e\}$.

Then, identifying the split vertices and edges, back to G , $(D, f_1 + f_2)$ is a 3-flow (D, f) with $\text{supp}(f) = E(G^2) \setminus \{e_0\}$. ■

Lemma 2.7 (1). *Let G be a k -path with $k \geq 2$ or an m -circuit with $m = 3$ or $m \geq 5$. Then G^2 admits a 3-NZF.*

(2). *Let G be a graph obtained from an r -circuit $x_0x_1 \cdots x_{r-1}x_0$ by attaching an edge x_iv_i at each x_i for $0 \leq i \leq r-1$, where $v_i \neq v_j$ if $i \neq j$. Then G^2 admits a 3-NZF.*

(3). *Let G be a graph obtained from an m -circuit $x_0x_1 \cdots x_{m-1}x_0$ by attaching an edge $x_{m-1}v$ at x_{m-1} alone, where $m \geq 5$. Then G^2 admits a 3-NZF.*

Proof. (1). If G is an m -circuit with $m = 3$ or $m \geq 5$, then G^2 is a cycle (every vertex is of even degree) and G^2 admits 2-NZF. If G is a k -path with $k \geq 2$, by induction on k and using Lemma 2.5-(2), G^2 admits a 3-NZF.

(2). For $r \geq 5$ (or $r = 3$): let D be an orientation such that v_i ($0 \leq i \leq r-1$) is the tail of every edge of G^2 incident with it and all the other edges are oriented as

$x_i \rightarrow x_{i+1}$, $x_i \rightarrow x_{i+2} \pmod r$ (or $x_i \rightarrow x_{i+1} \pmod 3$ only for $r = 3$). Obviously, D is a modular 3-orientation of G^2 .

For $r = 4$: let D be the orientation such that v_0 and v_2 be the tail of every edge of G^2 incident with it, v_1 and v_3 be the head of every edge of G^2 incident with it, $x_0x_1x_3x_2x_0$ as a directed circuit and other edges are oriented as $x_3 \rightarrow x_0$, $x_1 \rightarrow x_2$. Obviously, D is a modular 3-orientation of G^2 .

(3). Orient all the edges as $x_i \rightarrow x_{i+1}$, $x_i \rightarrow x_{i+2} \pmod m$ for $0 \leq i \leq m - 1$ and let v be the tail of every edge of G^2 incident with it. Then reverse the direction of the following edges: x_0x_{m-1}, x_0x_{m-2} . Clearly, this orientation is a modular 3-orientation of G^2 . ■

3 Proof of the main theorem

Proof. \implies By contradiction. Suppose $G \in \bar{\mathcal{T}}_{1,3}$. Let G be a counterexample with $|V(G)| + |E(G)|$ as small as possible. Clearly $|V(G)| \geq 5$ and G contains no circuits. So $G \in \mathcal{T}_{1,3}$. Let $v \in V(G)$ be a degree 3 vertex such that $N_G(v) = \{v_1, v_2, v_3\}$, $d_G(v_1) = d_G(v_2) = 1$. Clearly, $G_1 = G \setminus \{v_1, v_2\} \in \mathcal{T}_{1,3}$. Since G^2 has a modular 3-orientation D and both v_1 and v_2 are degree 3 vertices in G^2 , then this orientation restricted to the edge set of G_1^2 will generate a modular 3-orientation of G_1^2 . Therefore, G_1^2 admits a 3-NZF, a contradiction.

\Leftarrow Let G be a counterexample to the theorem such that

- (i). $|E(G)| - |V(G)|$ is as small as possible,
- (ii). subject to (i), $|E(G)|$ is as small as possible.

Note that $|E(G)| - |V(G)| + 1$ is the rank of the cycle space of G .

Claim 1. Let $e_0 = xy \in E(G)$. If $d_G(x) \geq 3$ and $d_G(y) \geq 2$, then xy is not a cut edge of G .

If e_0 is a cut-edge, then at least one component of G_{*e_0} is not in $\bar{\mathcal{T}}_{1,3}$, say, G_1 is not, while G_2 might be. By induction, let (D, f_i) be a 3-flow of G_i^2 for each $i = 1, 2$ such that f_1 is nowhere-zero, f_2 might miss only one edge e_x (that is a copy of e_0). Without loss of generality, assume that $f_1(e_y) + f_2(e_x) \not\equiv 0 \pmod 3$. Then, identifying the split vertices and edges, back to G , $(D, f_1 + f_2)$ is a nowhere-zero Modular 3-flow of G^2 . By Lemma 2.3, G^2 admits a 3-NZF, a contradiction.

Claim 2. $d_G(x) \leq 3$ for any $x \in V(G)$.

Otherwise, assume that $d_G(x) \geq 4$ for some vertex $x \in V(G)$. Clearly $G \not\cong K_{1,m}$ for $m \geq 4$ since $K_{1,m}$ is not a counterexample. So there exists $e_0 = xy \in E(G)$ with $d_G(y) \geq 2$. By Claim 1, e_0 is not a cut edge of G and $G_1 = G_{*e_0} \notin \bar{\mathcal{T}}_{1,3}$. Then by (i), G_1^2 admits a 3-NZF.

In G_1^2 , identify x and x' , y and y' , and use one edge to replace two parallel edges, by Lemma 2.3, we will get G^2 and a Modular 3-flow (D, f) of G^2 such that $E(G^2) \setminus \text{supp}(f) \subseteq \{xv \text{ or } yw \mid v \in N_G(y), w \in N_G(x)\}$. Let $C(x) = G^2[N_G(x) \cup \{x\}]$. Then $C(x)$ is a clique of order at least 5. We are to adjust (D, f) so that the resulting Modular 3-flow (D, f')

of G^2 misses only edges of $\{uv \mid u, v \in V(C(x))\}$. For each edge xv which is missed by $\text{supp}(f)$ and $xv \notin E(C(x))$, $xyvx$ must be a circuit of G^2 , so let (D, f_{xv}) be a 3-flow of G^2 with $\text{supp}(f_{xv}) = \{xy, yv, xv\}$ and $f_{xv}(yv) + f(yv) \not\equiv 0 \pmod{3}$. Now $(D, f + f_{xv})$ is a Modular 3-flow of G^2 whose support contains xv, yv , but may miss xy . Repeat this adjustment and do the similar adjustment for the edges yw not in the support until we get a Modular 3-flow (D, f') of G^2 such that $E(G^2) \setminus \text{supp}(f') \subseteq E(C(x))$. Since each edge in $C(x)$ is contained in some K_5 and thus is a center edge in some W_4 , by Lemma 2.3 and Lemma 2.5-(1), G^2 admits a 3-NZF, a contradiction.

Claim 3. No degree 2 vertex is contained in a 3-circuit.

By contradiction. Assume $xyzx$ is a circuit of G with $d_G(x) = 2$. If $d_G(y) = 2$, then we must have $d_G(z) = 3$. Therefore $G_1 = G \setminus \{xy\} \notin \bar{T}_{1,3}$ and $G_1^2 = G^2$, contradicting (ii). So $d_G(y) = d_G(z) = 3$.

Let $N_G(y) = \{x, y', z\}$ and $N_G(z) = \{x, y, z'\}$. Let $G_1 = G - \{x\}$. Since $(N_G(y) \cap N_G(z)) \setminus \{x\} = \emptyset$ (otherwise, let $G_2 = G \setminus \{yz\}$, then $G_2^2 = G^2$, $G_2 \notin \bar{T}_{1,3}$, contradicting (ii)) and $d_{G_1}(y) = 2$, then $G_1 \notin \bar{T}_{1,3}$. So G_1^2 admits a 3-NZF. Since $E(G^2) \setminus E(G_1^2) = \{xy, xy', xz, xz'\}$, by Lemma 2.5-(2), G^2 admits a 3-NZF, a contradiction.

Claim 4. No degree 2 vertex of G is contained in a 4-circuit.

Assume $C = xu_1u_2u_3x$ is a 4-circuit of G and $d_G(x) = 2$. By Claim 3, $u_1u_3 \notin E(G)$. Let u'_i be the adjacent vertex of u_i which is not in $V(C)$ if $d_G(u_i) = 3$ for some $i \in \{1, 2, 3\}$. Let $G_1 = G \setminus \{x\}$. We consider the following 3 cases.

Case 1. $d_G(u_1) = d_G(u_3) = 2$.

Then $d_G(u_2) = 3$ and $d_G(u'_2) \geq 2$ (if $d_G(u'_2) = 1$, it's easy to show G^2 admits a 3-NZF). Clearly, $u_2u'_2$ is a cut edge, contradicting Claim 1.

Case 2. Exactly one of u_1, u_3 has degree 3.

Assume $d_G(u_1) = 3$ and $d_G(u_3) = 2$. Since $d_{G_1}(u_1) = 2$, if $d_{G_1}(u'_1) = 2$ then u'_1 is not contained in a 3-circuit in G (by Claim 3), and so $G_1 \notin \bar{T}_{1,3}$. By induction, G_1^2 admits a 3-NZF. Since $E(G^2) \setminus E(G_1^2) = \{xu'_1, xu_1, xu_2, xu_3\}$ and $G^2[V(C) \cup \{u'_1\}]$ contains a W_4 with x as its center, by Lemma 2.5-(1), G^2 admits a 3-NZF, a contradiction.

Case 3. $d_G(u_1) = d_G(u_3) = 3$.

If $u'_1 = u'_3$, then $u'_1u_1u_2u_3$ is a 3-path, otherwise $u'_1u_1u_2u_3u'_3$ is 4-path. In both cases G_1^2 admits a 3-NZF. Since $E(G^2) \setminus E(G_1^2) = \{xu'_1, xu_1, xu_2, xu_3, xu'_3\}$ and each edge xu_i or xu'_j is contained in some W_4 in G^2 as a center edge for $1 \leq i \leq 3$ and $j = 1, 3$, by Lemma 2.5-(1), G^2 admits a 3-NZF. a contradiction.

Claim 5. For any $v \in V(G)$, $d_G(v) \neq 2$.

Otherwise, if there exists $v \in V(G)$ such that $d_G(v) = 2$, then by Claim 3-4, v is not contained in any circuits of length 3 or 4. By Lemma 2.7-(1), G cannot be a k -path with $k \geq 2$ or an m -circuit with $m = 3$ or $m \geq 5$. Let us consider the following cases.

Case 1. There exists a path $P_m = v_1v_2 \cdots v_m$ such that $m \geq 3$, $v = v_t$ for some $2 \leq t \leq m - 1$, $d_G(v_k) = 2$ for $2 \leq k \leq m - 1$ and $d_G(v_1) \neq 2$, $d_G(v_m) \neq 2$.

Clearly, at least one of v_1, v_m has degree 3. If $d_G(v_i) = 3$ for $i = 1$, or m , let $N_G(v_i) \setminus V(P_m) = \{v'_i, v''_i\}$. Clearly, $G_1 = G \setminus \{v_2, v_3, \dots, v_{m-1}\} \notin \bar{T}_{1,3}$ (because by

Claim 3, degree 2 vertices are not contained in any 3-circuits of G). By Claim 1, G_1 is connected. So G_1^2 admits a 3-NZF (D, f_1) . By Lemma 2.7-(1), P_m^2 admits a 3-NZF (D, f_2) . Then G^2 admits a 3-flow (D, f) with $\text{supp}(f) = \text{supp}(f_1) \cup \text{supp}(f_2)$. By Claim 3-4, $E(G^2) \setminus \text{supp}(f) = \{v_2v'_1, v_2v''_1, v_{m-1}v'_m, v_{m-1}v''_m\}$, then by Lemma 2.5-(2), G^2 admits a 3-NZF, a contradiction.

Case 2. There exists a m -circuit $C = v_1v_2 \cdots v_mv_1$ with $m \geq 5$, $d_G(v_i) = 2$ for $1 \leq i \leq m-1$, $d_G(v_m) = 3$ and $v = v_t$ for some $1 \leq t \leq m-1$.

Suppose that $v_0 \in N_G(v_m) \setminus V(C)$. By Claim 1, $d_G(v_0) = 1$. So by Lemma 2.7-(3), G^2 admits a 3-NZF, a contradiction.

Claim 6. Let $e = xy \in E(G)$ with $d_G(x) = d_G(y) = 3$. Then e is contained in a circuit of length 3 or 4.

By contradiction. Let G_1 be the graph obtained from G by deleting the edge e and adding a new vertex y' and a new edge xy' . Since G contains no degree 2 vertices and $d_{G_1}(y) = 2$, then $G_1 \notin \bar{T}_{1,3}$. By Claim 1, e is not a cut edge of G , then by (i), G_1^2 admits a 3-NZF (D, f_1) . Identify y and y' , the resulting 3-flow (D, f_2) in G^2 misses only two edges y_1x and y_2x where $N(y) = \{y_1, y_2, x\}$ (since xy is not contained a circuit of length 3 or 4). By Lemma 2.5-(2), G^2 admits a 3-NZF, a contradiction.

Claim 7. For each $x \in V(G)$ with $d_G(x) = 3$, $|N_G(x) \cap V_3| \leq 2$, where V_3 is the set of all the degree 3 vertices of G .

By contradiction. Assume that $U = \{u_1, u_2, u_3\} = N_G(x) \cap V_3$. Let $G_1 = G \setminus \{x\}$. By Claim 1, G_1 is connected. Since G contains no degree 2 vertices, $G_1 \notin \bar{T}_{1,3}$ and G_1^2 admits a 3-NZF (D, f) . By Claim 6, each xu_i ($1 \leq i \leq 3$) is contained a circuit of length at most 4. We consider the following 3 cases.

Case 1. $G[U]$ contains at least 2 edges.

Suppose that $u_1u_2, u_2u_3 \in E(G)$. Let $u'_i \in N_G(u_i) \setminus U$ for $i = 1, 3$. If $u'_1 = u'_3$, then $G^2[U \cup \{u'_1, x\}] \cong K_5$, by Lemma 2.5-(1), we can get a 3-NZF of G^2 , a contradiction. If $u'_1 \neq u'_3$, then $G[u'_1u_1u_2u_3u'_3]$ is a 4-path, by Lemma 2.5-(1) (similar to Case 3 of Claim 4), we can get a 3-NZF of G^2 , a contradiction.

Case 2. $G[U]$ contains exactly 1 edge.

Assume that $u_1u_2 \in E(G)$. By Claim 6, each edge xu_i ($i = 1, 2, 3$) is contained in a circuit of length 3 or 4. So we may assume $z \in (N_G(u_2) \cap N_G(u_3)) \setminus \{x\}$. Clearly, $G^* = G^2[U \cup \{x, z\}] \cong K_5$. Let $u'_i \in N_G(u_i) \setminus (U \cup \{z\})$ for $i = 1, 3$. Clearly, $E(G^2) \setminus \text{supp}(f) \subseteq E(G^*) \cup \{xu'_1, xu'_3\}$. Since $xu_ju'_jx$ ($j = 1, 3$) is a circuit of G^2 , we can get a 3-flow (D, f_1) such that $E(G^2) \setminus \text{supp}(f_1) \subseteq E(G^*)$. By Lemma 2.5-(1), we can get a 3-NZF of G^2 , a contradiction.

Case 3. $G[U]$ contains no edges.

Assume that $z_1 \in (N_G(u_1) \cap N_G(u_2)) \setminus \{x\}$ and $z_2 \in (N_G(u_1) \cap N_G(u_3)) \setminus \{x\}$. Let $G_2 = G \setminus \{xu_1\}$, then $G_2 \notin \bar{T}_{1,3}$ and G_2^2 admits a 3-NZF (D, f_1) . Clearly, $E(G^2) \setminus \text{supp}(f_1) = \{xu_1\}$. Since xu_1 is contained in a W_4 which is contained in the graph induced by $\{u_1, z_1, u_2, u_3, x\}$ in G^2 with x as center, by Lemma 2.5-(1), we can get a 3-NZF of G^2 , a contradiction.

Final Step. By Claim 2, Claim 5 and Claim 7, all vertices of G have degree 1 or 3 and each degree 3 vertex is adjacent to at most 2 degree 3 vertices. So $G[V_3]$ is a path or a circuit, hence G must be a graph obtained from an r -circuit $x_0x_1 \cdots x_{r-1}x_0$ by attaching an edge x_iv_i at each x_i for $0 \leq i \leq r-1$, where $v_i \neq v_j$ if $i \neq j$, or a path $x_0x_1 \cdots x_p$ by attaching an edge v_ix_i ($1 \leq i \leq p-1$) at each x_i , where $v_i \neq v_j$ if $i \neq j$. Clearly the latter case is a graph in $\bar{\mathcal{T}}_{1,3}$. By Lemma 2.7-(2), G^2 admits a 3-NZF, a contradiction. ■

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