

1994

# Suppression of Bremsstrahlung at nonzero temperature

HA Weldon

Follow this and additional works at: [https://researchrepository.wvu.edu/faculty\\_publications](https://researchrepository.wvu.edu/faculty_publications)

---

## Digital Commons Citation

Weldon, HA, "Suppression of Bremsstrahlung at nonzero temperature" (1994). *Faculty Scholarship*. 356.  
[https://researchrepository.wvu.edu/faculty\\_publications/356](https://researchrepository.wvu.edu/faculty_publications/356)

This Article is brought to you for free and open access by The Research Repository @ WVU. It has been accepted for inclusion in Faculty Scholarship by an authorized administrator of The Research Repository @ WVU. For more information, please contact [ian.harmon@mail.wvu.edu](mailto:ian.harmon@mail.wvu.edu).

# SUPPRESSION OF BREMSSTRAHLUNG AT NON-ZERO TEMPERATURE

H. Arthur Weldon

Department of Physics

West Virginia University

Morgantown, WV 26506-6315

July 26, 1993

The first-order bremsstrahlung emission spectrum is  $\alpha d\omega/\omega$  at zero temperature. If the radiation is emitted into a region that contains a thermal distribution of photons, then the rate is increased by a factor  $1 + N(\omega)$  where  $N(\omega)$  is the Bose-Einstein function. The stimulated emission changes the spectrum to  $\alpha T d\omega/\omega^2$  for  $\omega \ll T$ . If this were correct, an infinite amount of energy would be radiated in the low frequency modes. This unphysical result indicates a breakdown of perturbation theory. The paper computes the bremsstrahlung rate to all orders of perturbation theory, neglecting the recoil of the charged particle. When the perturbation series is summed, it has a different low-energy behavior. For  $\omega \ll \alpha T$ , the spectrum is independent of  $\omega$  and has a value proportional to  $d\omega/\alpha T$ .

arXiv:hep-ph/9308243v1 7 Aug 1993

# I. INTRODUCTION AND SUMMARY

## A. Background

In studies of the quark-gluon plasma to be produced in ultrarelativistic heavy ion collisions and, more generally, in studies of field theory at finite temperature, a central concern is how the cancellation of infrared divergences affects various finite, physical quantities. The experimentally measured rates for certain low energy processes can be significantly modified by the infrared structure.

This paper will investigate the emission of low energy photons by a charged particle that passes through a fixed-temperature plasma. The charged particle must undergo a collision in order to radiate. If the radiated energy is much smaller than the energy transfer in the collision then the inelastic cross section factors:

$$2\omega \frac{d\sigma}{d^3k dq^2} \approx 2\omega \frac{dP(q^2)}{d^3k} \frac{d\sigma}{dq^2} \quad (|k \cdot q| \ll |q^2|) \quad (1.1)$$

with the collision cross section  $d\sigma/dq^2$  independent of  $k \cdot q$ , where  $q = p - p'$ . Thus to investigate the probability of radiation,  $P$ , the details of the hard scattering do not matter. All that matters is that the charged particle began with four-momentum  $p^\mu$  and ended with four-momentum  $p'^\mu$ . The radiation probability to first order in  $\alpha$  is

$$2\omega \frac{dP_1}{d^3k} = \frac{1}{(2\pi)^3} \sum_{\text{pol}} |\epsilon_\mu \cdot J^\mu|^2 [1 + N(\omega)] \quad (1.2)$$

where  $\omega = |\vec{k}|$  and  $N(\omega) = [1 - \exp(\omega/T)]^{-1}$  is the Bose-Einstein function. When the radiated energy is small, the matrix element of the electromagnetic current is

$$J^\mu(k) = ie \left( \frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right) e^{-k/2\Lambda} \quad (1.3)$$

where  $\Lambda$  is a momentum-space regulator that will be necessary later. The current results from an on-shell charge of four-momentum  $p$  radiating an on-shell photon of four-momentum  $k$ , which gives a Feynman denominator  $(p-k)^2 - m^2 = -2p \cdot k$ . For low-energy radiation this current is valid regardless of the spin of the charged particle [1-3]. Because

of the denominators in (1.3) the radiation is mostly parallel to  $\vec{p}$  or  $\vec{p}'$ . When (1.2) is integrated over photon angles, the probability of radiating energy  $\omega$  in any direction is

$$\frac{dP_1}{d\omega} = \frac{A}{\omega} [1 + N(\omega)] e^{-\omega/2\Lambda} \quad (1.4)$$

The charged particle momenta only occur in the function  $A$ :

$$A(p \cdot p') = \frac{\alpha}{\pi} \left( \frac{1}{v} \ln\left(\frac{1+v}{1-v}\right) - 2 \right) \quad (1.5)$$

where  $v$  is defined by  $p \cdot p' = m^2(1 - v^2)^{-1/2}$ . In terms of momentum transfer  $Q$ , the limiting behaviors of  $A$  are [1,2]

$$A \rightarrow \frac{2\alpha}{3\pi} \frac{Q^2}{m^2} \quad \text{for } Q \ll m \quad (1.6a)$$

$$A \rightarrow \frac{2\alpha}{\pi} \ln\left(\frac{Q^2}{m^2}\right) \quad \text{for } Q \gg m \quad (1.6b)$$

Formula (1.4) is classical, except for photon quantum statistics which produce the factor  $N$ . The formula fails at low energies because it predicts that the total energy radiated will be infinite:

$$\int_0^{E_{\max}} d\omega \frac{dP_1}{d\omega} = \infty \quad (1.7)$$

regardless of the value of  $E_{\max}$ . This does not occur at zero temperature, where the radiation probability is  $A/\omega$ ; but at non-zero temperature the Bose-Einstein factor makes it more singular:

$$\frac{dP_1}{d\omega} \approx \frac{AT}{\omega^2} \quad (\omega \ll T) \quad (1.8)$$

The breakdown comes from the smallest values of  $\omega$ , where the classical current approximation (1.3) works best. The resolution of the breakdown must occur from a higher order calculation.

## B. Higher Orders

To go to higher orders in  $\alpha$  one must quantize the electromagnetic field with the interaction

$$\mathcal{L}_I = -J^\mu(x)A_\mu(x) \quad (1.9)$$

with  $J^\mu$  the classical current (1.3) and  $A_\mu$  the quantized field. This interaction will produce multiple emissions and absorptions of real photons as well as closed loops of virtual photons. For example, to second order in  $\alpha$  there are three types of contributions to the probability of radiating energy  $\omega$ : either radiate two photons whose energy totals  $\omega$ ; or radiate one photon of energy  $\omega + \omega'$  and absorb another whose energy is  $\omega'$ , giving a net energy of  $\omega$ ; or radiate a single photon of energy  $\omega$  with a one-loop correction. The infrared divergences cancel among these processes and give a finite answer  $dP_2/d\omega$  to order  $\alpha^2$ .

The semiclassical interaction (1.9) does not conserve energy-momentum: it allows the particle of momentum  $p^\mu$  to radiate and still have momentum  $p^\mu$  (provided the hard collision eventually deflects the charge to  $p'^\mu$ ). This is a sensible approximation if each photon energy (real or virtual) is much smaller than the energy of the charged particle. To maintain this consistently in higher orders it is convenient to introduce the momentum cutoff  $\Lambda$  in (1.3) and require  $\Lambda \ll (\vec{p}^2 + m^2)^{1/2}$  and  $\Lambda \ll (\vec{p}'^2 + m^2)^{1/2}$ .

Because the current is classical, one can compute the generating functional for the multi-photon Green functions by performing a Gaussian functional integral. This allows one to calculate the exact multi-photon amplitudes  $\mathcal{M}$ . The probability of radiating a net energy  $\omega$  to all orders in perturbation theory is

$$\frac{dP}{d\omega} = \sum_{n=1}^{\infty} \int d\Phi_1 \dots d\Phi_n \delta(k_1^0 + \dots k_n^0 - \omega) \frac{1}{n!} \sum_{\text{pol}} |\mathcal{M}(k_1, \dots k_n)|^2 \quad (1.10)$$

Positive  $k^0$ 's correspond to emission of photons; negative  $k^0$ 's correspond to absorption of

photons. The phase space integration includes the appropriate statistical factors:

$$d\Phi = \frac{d^3k}{2k(2\pi)^3} \times \begin{cases} 1 + N(k) & k^0 = +k \\ N(k) & k^0 = -k \end{cases} \quad (1.11)$$

The amplitudes  $\mathcal{M}$  contain infrared divergences from the virtual photon integrations, but the integration over the real photons in (1.10) produces an infrared-finite probability. This is the Bloch-Nordsieck cancellation at  $T \neq 0$ . The cancellation is very delicate: It would not occur if negative  $k^0$ 's (i.e. energy absorption from the heat bath) were omitted from the delta function in (1.10).

### C. Summary of Results

Remarkably, one can calculate the probability (1.10) with only the approximation  $T \ll \Lambda$ . The final answer obtained at the end of Sec. III is

$$\frac{dP}{d\omega} = \left| \Gamma\left(\frac{A}{2} + i\frac{\omega}{2\pi T}\right) \right|^2 \frac{e^{\omega/2T} e^{-|\omega|/\Lambda}}{4\pi^2 T \Gamma(A)} \left(\frac{2\pi T}{\Lambda}\right)^A \quad (1.12)$$

Here  $A \propto \alpha$  is the same function of  $p \cdot p'$  as (1.5). This result also applies if  $\omega$  is negative, which means that the total emitted energy is less than the total absorbed energy. For  $\omega = -\tilde{\omega} < 0$ , it has the property

$$\frac{dP}{d\omega} = e^{-\tilde{\omega}/T} \frac{dP}{d\tilde{\omega}} \quad (1.13)$$

In the first order result (1.4), this corresponds to  $\exp(-\tilde{\omega}/T)[1 + N(\tilde{\omega})] = N(\tilde{\omega})$ . The zero temperature limit of (1.12) is discussed in Appendix A.

At  $\omega = 0$  the exact result (1.12) is finite. This resolves the total energy problem that arose in (1.7). The most interesting feature of (1.12) is that it involves two dimensionless quantities:  $A$  and  $\omega/T$ . To display a simpler form it is helpful to assume  $A \ll 1$ . This is quite reasonable because extremely large momentum transfers are necessary to make  $A$  large (for example,  $A = 1/16$  requires  $Q/m = 10^3$ ). Even with  $A \ll 1$  the behavior of (1.12) depends on the size of  $\omega/T$ . If  $A\pi \ll \omega/T$  then

$$\frac{dP}{d\omega} \approx \frac{A}{\omega} [1 + N(\omega)] e^{-\omega/\Lambda} \quad (A\pi T \ll \omega) \quad (1.14)$$

which agrees with (1.4). However this does not hold at the smallest values of  $\omega$ . To approximate (1.12) for  $A \ll 1$  and  $\omega \ll T$  one can use  $\Gamma(z) \approx 1/z$  when  $|z| \ll 1$ . Then

$$\frac{dP}{d\omega} \approx \frac{AT}{\omega^2 + (A\pi T)^2} \quad (\omega \ll T) \quad (1.15)$$

Naturally (1.15) agrees with (1.14) in the region of overlap,  $A\pi T \ll \omega \ll T$ . But for  $\omega \ll A\pi T$  the radiation is suppressed relative to the first order rate. Fig. 1 compares (1.12) with the first-order result over a wide range of  $\omega$ .

Naturally the net radiated energy is finite,

$$\int_0^T d\omega \omega \frac{dP}{d\omega} \approx AT \tan^{-1}\left(\frac{1}{A\pi}\right) \quad (1.16)$$

and numerically small. In the denominator of (1.15) the quantity  $A\pi T$  appears to be some type of thermal mass of order  $e^2 T$ . However, the following discussion will argue that  $A\pi T = \Gamma/2$ , where  $\Gamma$  is the radiation-damping rate.

It is necessary to emphasize several points: (i) At  $T = 0$  the exact bremsstrahlung probability does not enjoy this damping. It is peculiar to  $T \neq 0$ . (ii) Whenever  $T > 0$  the suppression will occur for a charge of any energy. Whether the charge is relativistic or nonrelativistic alters the value of  $A$ , but the plateau shown in Fig. 1 will always exist.

#### D. Interpretation of the Suppression

One can understand the suppression of low frequency bremsstrahlung as the radiation reaction brought about by unitarity. First, we examine the failure of the first-order bremsstrahlung probability by writing it as

$$\begin{aligned} \frac{dP_1}{d\omega} &= \frac{|F_1(\omega)|^2}{2\pi} \\ F_1(\omega) &= \sqrt{2\pi A} \left( \frac{1 + N(\omega)}{\omega} \right)^{1/2} e^{-|\omega|/2\Lambda} \end{aligned} \quad (1.17)$$

Let the Fourier transform of this be

$$\tilde{F}_1(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} F_1(\omega) \quad (1.18)$$

Then the bremsstrahlung probability has a time-dependence

$$\frac{dP_1}{dt} = |\tilde{F}_1(t)|^2 \quad (1.19)$$

As  $t \rightarrow \infty$  the Fourier transform of (1.17) behaves like

$$\tilde{F}_1(t) \rightarrow -i\sqrt{\Gamma} \quad (1.20)$$

where  $\Gamma \equiv 2\pi AT$ . Therefore  $dP_1/dt \rightarrow \Gamma$  as  $t \rightarrow \infty$ . It is obviously unphysical for the radiation probability to remain constant after an infinitely long time. This disease is familiar in elementary atomic physics calculations of radiative transitions (e.g.  $2p \rightarrow 1s + \gamma$  in hydrogen). The resolution is that at higher orders the transition amplitude usually falls exponentially with time. Hence a reasonable guess is that higher order calculations should replace (1.20) by

$$\tilde{F}(t) \stackrel{?}{=} -i\theta(t)\sqrt{\Gamma} e^{-\Gamma t/2} \quad (1.21)$$

The Fourier transform to frequency gives

$$F(\omega) \stackrel{?}{=} \frac{\sqrt{\Gamma}}{\omega + i\Gamma/2} \quad (1.22)$$

The guess gives a radiation probability

$$\frac{dP}{d\omega} \stackrel{?}{=} \frac{1}{2\pi} \frac{\Gamma}{\omega^2 + (\Gamma/2)^2} \quad (1.23)$$

which coincides with (1.15) the weak-coupling, low-energy limit. Note that this radiation damping does not come from modifications to the charged particle trajectory.

The remainder of the paper describes the calculation that yields (1.12). Sec. II formulates the bremsstrahlung problem in detail and Sec. III performs the necessary integrations. Appendix A discusses the  $T = 0$  limit of (1.12) and its relation to conventional  $T = 0$  calculations. Appendix B discusses the relation of (1.12) to some previous work of mine.



## II. THERMAL BREMSSTRAHLUNG

When charged particles are described by classical currents  $J^\mu$ , quantizing the radiation field becomes elementary. The generating functional for multi-photon Green functions is

$$Z(J) = \int D[A] \exp[i \int_C d^4x (\mathcal{L}_0 - J \cdot A)] \quad (2.1)$$

where  $C$  is a contour in the complex  $x^0$  plane that incorporates the temperature. For a real-time formulation [4-6] this gives

$$Z(J) = \exp\left[\frac{-i}{2} \int \frac{d^4k}{(2\pi)^4} J^\mu(k) D_{\mu\nu}(k) J^\nu(-k)\right] \quad (2.2)$$

where the finite-temperature propagator is

$$D_{\mu\nu}(k) = -g_{\mu\nu} \left( \frac{1}{k^2 + i\epsilon} - i2\pi\delta(k^2)N \right) \quad (2.3)$$

with  $N = [\exp(|\vec{k}|/T) - 1]^{-1}$ . The amplitude for one real photon is

$$\begin{aligned} \mathcal{M}(k) &= -ik^2 \epsilon_\mu(k) \frac{\delta Z(J)}{\delta J^\mu(-k)} (2\pi)^4 \\ &= \epsilon_\mu(k) J^\mu(k) Z(J) \end{aligned} \quad (2.4)$$

This amplitude describes emission if  $k^0 > 0$ ; absorption, if  $k^0 < 0$ . The amplitude for  $n$  photons is

$$\mathcal{M}(k_1, k_2, \dots, k_n) = Z(J) \prod_{\ell=1}^n [\epsilon_\mu(k_\ell) J^\mu(k_\ell)] \quad (2.5)$$

To compute probabilities, one must square the amplitude and integrate over photon momenta. For the photon phase space it is convenient to employ the notation

$$d\Phi_\ell = \frac{d^4k_\ell}{(2\pi)^3} \delta(k_\ell^2) [\theta(k_\ell^0) + N(|\vec{k}_\ell|)] \quad (2.6)$$

This correctly weights photon emission with the statistical factor  $1 + N$ ; and photon absorption with the factor  $N$ . The probability of radiating a net energy  $\omega$  is

$$\begin{aligned} \frac{dP}{d\omega} &= \sum_{n=1}^{\infty} \int d\Phi_1 \dots d\Phi_n \delta(k_1^0 + \dots k_n^0 - \omega) \\ &\quad \frac{1}{n!} \sum_{\text{pol}} |\mathcal{M}(k_1, \dots, k_n)|^2 \end{aligned} \quad (2.7)$$

The polarization sums give

$$\sum_{\text{pol}} |\mathcal{M}(k_1, \dots, k_n)|^2 = |Z(J)|^2 \prod_{\ell=1}^n J_\mu(k_\ell) J^\mu(k_\ell) \quad (2.8)$$

It is important to note that the current (1.3) has the properties

$$\begin{aligned} J_\mu(k) J^{\mu*}(k) &= J_\mu(k) J^\mu(-k) \\ &= -J_\mu(k) J^\mu(k) < 0 \end{aligned} \quad (2.9)$$

*Virtual Photons:* The integration over all virtual photons is contained in the multiplicative factor  $|Z(J)|^2$ . Using (2.2) and (2.3) gives

$$\begin{aligned} |Z(J)|^2 &= \exp(V) \\ V &= - \int \frac{d^3 k}{2k(2\pi)^3} J_\mu(k) J^\mu(k) [1 + 2N] \end{aligned} \quad (2.10)$$

Because  $J \sim 1/k$ , this integral contains both linear and logarithmic divergences in the infrared.

*Real Photons:* The delta function that constrains the real photon energies can be represented as

$$\delta(k_1^0 + \dots k_n^0 - \omega) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-i\omega z} e^{i(k_1^0 + \dots k_n^0)z} \quad (2.11)$$

In (2.7) each integration over  $d^4 k_\ell$  will give the same function of  $z$ :

$$R(z) = \int d\Phi e^{ik^0 z} \sum_{\text{pol}} |\epsilon_\mu(k) J^\mu(k)|^2 \quad (2.12)$$

More explicitly this is

$$\begin{aligned} R(z) &= \int \frac{d^3 k}{2k(2\pi)^3} J_\mu(k) J^\mu(k) \\ &\quad \times ([1 + N] e^{ikz} + N e^{-ikz}) \end{aligned} \quad (2.13)$$

This is the contribution of the real photons. Because  $J \sim 1/k$ , this integral has both linear and logarithmic infrared divergence.

The probability (2.7) is

$$\begin{aligned} \frac{dP}{d\omega} &= |Z(J)|^2 \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-i\omega z} \sum_{n=1}^{\infty} \frac{[R(z)]^n}{n!} \\ &= \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-i\omega z} \exp[\bar{R}(z)] \end{aligned} \quad (2.14)$$

(the  $n = 0$  term has no Fourier transform at  $\omega \neq 0$ ) and  $\bar{R}$  is

$$\begin{aligned} \bar{R}(z) &= R(z) + V \\ &= \int \frac{d^3k}{2k(2\pi)^3} J_{\mu}(k) J^{\mu}(k) \\ &\quad \times ([1 + N]e^{ikz} + Ne^{-ikz} - [1 + 2N]) \end{aligned} \quad (2.15)$$

Each term in  $\bar{R}$  has a clear interpretation. Recall that  $z$  is the variable conjugate to the net energy  $\omega$ . In (2.15) the term proportional to  $1 + N$  represents the stimulated emission of energy; the term proportional to  $N$  represents the absorption of energy; the subtracted term  $1 + 2N$  represents all virtual photons (emitted and absorbed). At small  $k$ ,  $J^{\mu} \sim 1/k$  and  $N \sim 1/k$ . Expanding out the  $\exp(\pm ikz)$  for small  $k$  shows that the linearly divergent terms  $dk/k^2$  cancel and the logarithmically divergent terms  $dk/k$  cancel. Thus  $\bar{R}$  is a completely finite function. (See (3.15) for  $\bar{R}$ .)

### III. EXPLICIT INTEGRATION

#### A. Angular Integration

The remaining task is to perform the integrations necessary for the probability (2.14).

The first step is to write

$$\begin{aligned} \bar{R}(z) &= \int_0^{\infty} \frac{dk}{k} A \\ &\quad \times ([1 + N]e^{ikz} + e^{-ikz} - [1 + 2N]) \end{aligned} \quad (3.1)$$

where  $A$  contains the angular integration:

$$A = \frac{k^2}{2} \int \frac{d\Omega}{(2\pi)^3} J_{\mu}(k) J^{\mu}(k) \quad (3.2)$$

Using  $k^\mu = k(1, \hat{k})$  in the current (1.3) gives

$$J_\mu(k)J^\mu(k) = \frac{e^2}{k^2} \left[ \frac{2(E'E - \vec{p}' \cdot \vec{p})}{(E' - \vec{p}' \cdot \hat{k})(E - \vec{p} \cdot \hat{k})} - \frac{m^2}{(E' - \vec{p}' \cdot \hat{k})^2} - \frac{m^2}{(E - \vec{p} \cdot \hat{k})^2} \right] e^{-k/\Lambda} \quad (3.3)$$

The factors of  $k$  completely cancel so that  $A$  is independent of  $k$ . This is the same angular integration that arises in lowest order bremsstrahlung and is described in textbooks [2]. It turns out that  $A$  is a rather messy function of momentum transfer  $Q$ . It is a bit simpler to express it in terms of  $\sigma = p' \cdot p/m^2$ :

$$A = \frac{2\alpha}{\pi} \left( \frac{\sigma}{\sqrt{\sigma^2 - 1}} \ln(\sigma + \sqrt{\sigma^2 - 1}) - 1 \right) \quad (3.4)$$

As noted by Weinberg [7], it is simplest to express  $A$  in terms of the relative velocity  $v$  of the final particle in the rest frame of the initial particle (or vice versa) defined by  $p' \cdot p = m^2(1 - v^2)^{-1/2}$  so that

$$A(p \cdot p') = \frac{\alpha}{\pi} \left( \frac{1}{v} \ln\left(\frac{1+v}{1-v}\right) - 2 \right) \quad (3.5)$$

## B. Integration Over $k$

To perform the  $k$  integration necessary for (3.1), separate  $\bar{R}$  into a temperature-independent part and a temperature-dependent part:

$$\bar{R}(z) = \bar{R}_0(z) + \bar{R}_T(z) \quad (3.6a)$$

$$\bar{R}_0(z) = A \int_0^\infty \frac{dk}{k} (e^{ikz} - 1) \exp(-k/\Lambda) \quad (3.6b)$$

$$\bar{R}_T(z) = 2A \int_0^\infty \frac{dk}{k} \frac{\cos(kz) - 1}{\exp(k/T) - 1} \exp(-k/\Lambda) \quad (3.6c)$$

Both  $\bar{R}_0$  and  $\bar{R}_T$  are infrared finite. For  $\bar{R}_0(z)$ , expand the integrand in powers of  $z$

$$\bar{R}_0(z) = A \sum_{n=1}^{\infty} \frac{(iz)^n}{n!} \int_0^\infty dk k^{n-1} e^{-k/\Lambda} \quad (3.7)$$

The displayed integrals over  $k$  each give  $(n-1)!\Lambda^n$ . The sum on  $n$  is elementary:

$$\bar{R}_0(y) = -A \ln[1 - i\Lambda z] \quad (3.8)$$

For  $\bar{R}_T(z)$ , put  $k = 2Tx$  so that

$$\bar{R}_T(z) = A \int_0^\infty \frac{dx}{x} \frac{\cos(2Tzx) - 1}{\sinh(x)} e^{-x[1+2T/\Lambda]} \quad (3.9)$$

This is finite at  $x = 0$  but it is convenient to multiply the integrand by an additional convergence factor  $x^\mu$  in order to use [8]

$$\int_0^\infty dx x^{\mu-1} \frac{e^{-\beta x}}{\sinh(x)} = 2^{1-\mu} \Gamma(\mu) \zeta\left[\mu, \frac{\beta+1}{2}\right] \quad (3.10)$$

This gives

$$\bar{R}_T(z) = \lim_{\mu \rightarrow 0} A 2^{1-\mu} \Gamma(\mu) \operatorname{Re} \left( \zeta[\mu, \hat{q}] - \zeta[\mu, q] \right) \quad (3.11)$$

where  $\hat{q} = q + iTz$  and  $q = 1 + T/\Lambda$ . Now expand  $\zeta$  in a Taylor series about  $\mu = 0$  using [8]

$$\zeta[\mu, q]|_{\mu=0} = -q + 1/2 \quad (3.12a)$$

$$\frac{d\zeta[\mu, q]}{d\mu}|_{\mu=0} = \ln[\Gamma(q)] - \frac{1}{2} \ln(2\pi) \quad (3.12b)$$

When the limit  $\mu \rightarrow 0$  is taken in (3.11) the result is

$$\bar{R}_T(z) = A \ln \left( \frac{\Gamma(q + iTz)\Gamma(q - iTz)}{\Gamma(q)\Gamma(q)} \right) \quad (3.13)$$

$\bar{R}(z)$  is the sum of (3.8) and (3.13).

### C. Integration Over $z$

The last integral to perform is the Fourier transform

$$\frac{dP}{d\omega} = \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-i\omega z} \exp(\bar{R}(z)) \quad (3.14)$$

where the integrand is

$$\exp(\bar{R}(z)) = \left[ \frac{\Gamma(q + iTz)\Gamma(q - iTz)}{(1 - i\Lambda z)\Gamma(q)\Gamma(q)} \right]^A \quad (3.15)$$

and  $q = 1 + T/\Lambda$ . Since  $A$  is non-integer, (3.15) has an infinite number of branch cuts in the variable  $z$ . For  $\omega > 0$  (corresponding to net emission of energy by the charge) one can close the  $z$  contour in the lower half plane. Then there is a branch point at  $z_0 = -i/\Lambda$  and an infinite set of branch points at  $z_n = -i/\Lambda - in/T$  for  $n=1,2,3\dots$  that come from the poles of  $\Gamma(q - iTz)$ . Choose the branch cuts to run from  $z_n$  to  $z_n + \infty$  as shown in Fig. 2 so that

$$\frac{dP}{d\omega} = \sum_{n=0}^{\infty} \int_{z_n}^{z_n + \infty} \frac{dz}{2\pi} e^{-i\omega z} \text{Disc}_n \left[ \exp(\bar{R}(z)) \right] \quad (3.16)$$

The discontinuity across the  $n$ 'th branch cut is

$$\text{Disc}_n \left[ \exp(\bar{R}(z)) \right] = [1 - e^{-i2\pi A}] \exp(\bar{R}(z)) \quad (3.17)$$

Then put  $z = z_n + r$  where  $r$  is real:

$$\begin{aligned} \frac{dP}{d\omega} &= [1 - e^{-i2\pi A}] \sum_{n=0}^{\infty} e^{-i\omega z_n} \\ &\int_0^{\infty} \frac{dr}{2\pi} e^{-i\omega r} \exp(\bar{R}(z_n + r)) \end{aligned} \quad (3.18)$$

The value of the function along the branch cut is

$$\exp(\bar{R}(z_n + r)) = \left[ C_n(r) \frac{\pi T e^{i\pi(-n+1/2)}}{\Lambda \sinh(\pi T r)} \right]^A \quad (3.19a)$$

$$C_n(r) = \frac{\Gamma(1 + 2T/\Lambda + n + iT r)}{\Gamma(1 + n + iT r) [\Gamma(1 + T/\Lambda)]^2} \quad (3.19b)$$

At this stage it is necessary to make the approximation  $T \ll \Lambda$  so that  $C_n(r) \rightarrow 1$ . This allows the summation on  $n$  in (3.18) to be done:

$$\frac{dP}{d\omega} = \frac{e^{\omega/2T} e^{-\omega/\Lambda} \sin(A\pi)}{\sin(A\pi/2 - i\omega/2T)} \left( \frac{\pi T}{\Lambda} \right)^A \frac{I(\omega)}{2\pi} \quad (3.20)$$

$$I(\omega) = \int_0^{\infty} dr e^{-i\omega r} [\sinh(\pi T r)]^{-A} \quad (3.21)$$

This is a known integral [8]

$$I(\omega) = \frac{\Gamma(1 - A) \Gamma(A/2 + i\omega/2\pi T)}{\pi T 2^{1-A} \Gamma(1 - A/2 + i\omega/2\pi T)} \quad (3.22)$$

Using the reflection property of the gamma function gives the final result quoted in (1.12):

$$\frac{dP}{d\omega} = \frac{e^{\omega/2T} e^{-\omega/\Lambda}}{\Gamma(A)4\pi^2T} \left(\frac{2\pi T}{\Lambda}\right)^A \left|\Gamma\left(\frac{A}{2} + i\frac{\omega}{2\pi T}\right)\right|^2 \quad (3.23)$$

#### ACKNOWLEDGMENT

This work was supported in part by the U.S. National Science Foundation under grant PHY-9213734.

#### APPENDIX: THE ZERO-TEMPERATURE LIMIT

At zero temperature (1.12) has the simple behavior

$$\frac{dP}{d\omega} = \frac{1}{\omega} \left(\frac{\omega}{\Lambda}\right)^A \frac{e^{-\omega/\Lambda}}{\Gamma(A)} \quad (A1)$$

and easily satisfies

$$\int_0^\infty d\omega \frac{dP}{d\omega} = 1 \quad (A2)$$

Since all detectors have some energy threshold for detection, physical probabilities must include an integration over the below-threshold photons. If the detector measures total radiant energy deposition (as in a calorimeter) above  $E_{\min}$  then the appropriate probability is

$$\sum_{n=1}^\infty \int \prod_{\ell=1}^n \left[ \frac{d^3k_\ell}{2k_\ell(2\pi)^3} \right] \theta(E_{\min} - \sum_j k_j) \frac{|\mathcal{M}_n|^2}{n!} \quad (A3)$$

This probability is

$$\int_0^{E_{\min}} d\omega \frac{dP}{d\omega} = \left(\frac{E_{\min}}{\Lambda}\right)^A \frac{1}{\Gamma(1+A)} \quad (A4)$$

with  $\exp(-\omega/\Lambda)$  neglected.

The more familiar situation is a detector that is sensitive to single photons each of which has energy above some threshold  $E_{\min}$ . In that case the single theta function in (A3) is replaced by a product:

$$\theta(E_{\min} - k_1)\theta(E_{\min} - k_2)\dots\theta(E_{\min} - k_n) \quad (A4)$$

This calculation is discussed by Weinberg [7] and gives a probability

$$\left(\frac{E_{\min}}{\Lambda}\right)^A b(A) \quad (A6)$$

where  $b(A) \approx 1 - (\pi A)^2/12 + \dots$  is a different function than the  $1/\Gamma(1 + A)$  that occurs in (A4).

## APPENDIX B: COMMENT ON A PREVIOUS CALCULATION

In a previous paper on the semiclassical approximation [9], I computed the residual effects of infrared cancellation on processes which occur in a finite-volume heat bath but with no photons detected outside the heat bath. The absence of high energy photons means none were radiated by the charge. The absence of low energy photons, which have a short mean free path, is unavoidable because all those radiated by the charge would be thermalized in the heat bath and lose their identity. Consequently there is a threshold energy  $\epsilon$  that depends on the size of the system. In the notation of Sec. II, the photons with  $k < \epsilon$  contribute

$$R_\epsilon(z) = \int_0^\epsilon \frac{d^3k}{2k(2\pi)^3} J_\mu(k) J^\mu(k) \times ([1 + N]e^{ikz} + Ne^{-ikz}) \quad (B1)$$

There is no constraint on the total energy  $\omega$  so the probability of no detected photons is

$$\int_0^\infty d\omega \frac{dP_\epsilon}{d\omega} = \exp[\bar{R}_\epsilon(0)] \quad (B2)$$

where

$$\begin{aligned} \bar{R}_\epsilon(0) &= V + R_\epsilon(0) \\ &= -A \int_\epsilon^\infty \frac{dk}{k} [1 + 2N] e^{-k/\Lambda} \end{aligned} \quad (B3)$$

If  $\epsilon \ll T$  then  $\bar{R}_\epsilon(0) \approx -2AT/\epsilon$ . However, for reasonable mean free paths it invariably turns out that  $\epsilon > T$ , which made these thermal corrections small.



## REFERENCES

1. D.R. Yennie, S.C. Frautschi, and H. Suura, *Ann. of Phys. (NY)*, **13**, 379 (1961).
2. C. Itzykson and J.B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
3. S. Weinberg, *Phys. Rev.* **135**, B1049 (1965).
4. A.J. Niemi and G.W. Semenoff, *Ann. of Phys. (NY)* **152**, 105 (1984).
5. N.P.Landsman and Ch. G. van Weert, *Phys. Rep.* **145**, 141 (1987).
6. R. J. Rivers, *Path Integral Methods in Quantum Field Theory* (Cambridge Univ. Press, Cambridge Eng., 1987).
7. S.W. Weinberg, *Phys. Rev.* 140, B516 (1965).
8. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products*
9. H.A. Weldon, *Phys. Rev. D* 44, 3955 (1991); and *Hot Summer Daze: BNL Summer Study on QCD at Nonzero Temperature and Density*, ed. A. Gocksch and R. Pisarski (World Scientific, Singapore, 1992), p. 226. Academic Press, New York, 1980).

## FIGURE CAPTIONS

Fig. 1 The dimensionless probability,  $TdP/d\omega$ , for a charged particle to radiate a net energy  $\omega$ . The first order result fails when  $\omega < A\pi T$ . For  $\omega \ll A\pi T$  the exact result is constant,  $TdP/d\omega \approx 1/A\pi^2$ . In this plot  $A = 0.01$ , which corresponds to a momentum transfer  $Q/m = 3$ .

Fig. 2 Location of the branch cuts of the function  $\exp(\bar{R}(z))$  at  $z = -i/\Lambda - in/T$ . For  $\omega > 0$  the  $z$  integration contour may be wrapped around the branch cuts in the lower half-plane.