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# Lipschitz Restrictions of Continuous Functions and a Simple Construction of Ulam-Zahorski C1 Interpolation

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# LIPSCHITZ RESTRICTIONS OF CONTINUOUS FUNCTIONS AND A SIMPLE CONSTRUCTION OF ULAM-ZAHORSKI $C^1$ INTERPOLATION

**Abstract**

We present a simple argument that for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ its restriction to some perfect set is Lipschitz. We will use this result to provide an elementary proof of the $C^1$ free interpolation theorem, that for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ there exists a continuously differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ which agrees with $f$ on an uncountable set. The key novelty of our presentation is that no part of it, including the cited results, requires from the reader any prior familiarity with the Lebesgue measure theory.

## 1 Introduction and background

The main result we like to discuss here is the following 1985 theorem of Agronsky, Bruckner, Laczkovich, and Preiss [1]. It implies that every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ must have some traces of differentiability, even though there exist continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that are nowhere differentiable (see e.g. [10, 22, 23]) or, even stronger, nowhere approximately and $\mathcal{I}$-approximately differentiable. In fact, the first coordinate of the classical Peano curve (i.e., $f_1\colon [0,1] \to [0,1]$, where $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ is a continuous surjection constructed by Peano) has these properties, see [6] or

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[7, Example 4.3.8]. Such a function cannot agree with a $C^1$ function on a set which is either of second category or of positive Lebesgue measure.

**Theorem 1.** *For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is a continuously differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ such that the set $[f = g] = \{x \in \mathbb{R}\colon f(x) = g(x)\}$ is uncountable. In particular, $[f = g]$ contains a perfect set $P$ and the restriction $f \restriction P$ is continuously differentiable.*

In the statement of Theorem 1 the differentiability of $h = f \restriction P$ is understood as the existence of its derivative, that is, of the function $h'\colon P \to \mathbb{R}$ defined, for every $p \in P$, as $h'(p) = \lim_{x\to p,\ x\in P} \frac{h(x)-h(p)}{x-p}$.

The story behind Theorem 1 spreads over a big part of the 20th century and is described in detail in [2] and [16]. Briefly, around 1940 S. Ulam asked, in Scottish Book, Problem 17.1, see [21], whether every continuous $f\colon \mathbb{R} \to \mathbb{R}$ agrees with some real analytic function on an uncountable set. Z. Zahorski showed, in his 1948 paper [25], that the answer is no: there exists a $C^\infty$ (i.e., infinitely many times differentiable) function which can agree with every real analytic function on at most finite set of points. At the same paper Zahorski stated a problem, refereed to as Ulam-Zahorski problem: does every continuous $f\colon \mathbb{R} \to \mathbb{R}$ agrees with some $C^\infty$ (or possibly $C^n$ or $D^n$) function on some uncountable set? Clearly, Theorem 1 shows that Ulam-Zahorski problem has an affirmative answer for the $C^1$ class of functions. This is the best possible result in this direction, since A. Olevskiĭ constructed, in his 1994 paper [16], a continuous function which can agree with every $C^2$ function on at most countable set of points.

The format of our proof of Theorem 1 is relatively straightforward. First we provide a simple argument that for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ its restriction to some perfect set $P \subset \mathbb{R}$ is Lipschitz.[1] Here the key case, presented in Sec. 2, is when $f$ is monotone. Then we will follow an argument of Morayne [15] to show that there is a perfect $Q \subset P$ for which $f \restriction Q$ satisfies the assumptions of Whitney's $C^1$ extension theorem [24]. At this point, to make the argument more accessible, we point the reader to a version of Whitney's $C^1$ extension theorem from [4], whose proof is elementary and simple.

[1] Of course this result follows immediately from Theorem 1, as $g$ from Theorem 1 is Lipschitz on any bounded interval. However, we are after a simpler proof of Theorem 1, so using it to argue for our step to prove it is pointless.

## 2 Lipschitz restrictions of monotone continuous maps

In what follows $f$ will always be a continuous function from $\mathbb{R}$ into $\mathbb{R}$, $\Delta$ will stand for the set $\{\langle x,x\rangle\colon x\in\mathbb{R}\}$, and $q\colon\mathbb{R}^2\setminus\Delta\to\mathbb{R}$ be the quotient function for $f$, that is, defined as $q(x,y)=\frac{f(x)-f(y)}{x-y}$. For $Q\subset\mathbb{R}$ we will use the symbol $q\restriction Q^2$ to denote the restriction of $q$ to the set $Q^2\setminus\Delta$.

**Theorem 2.** *Assume that $f\colon\mathbb{R}\to\mathbb{R}$ is monotone and continuous on a non-trivial interval $[a,b]$. For every $L>|q(a,b)|$ there exists a closed uncountable set $P\subset[a,b]$ such that $f\restriction P$ is Lipschitz with constant $L$.*

The difficulty in proving Theorem 2 without measure theoretical tools comes from the fact that there exist strictly increasing continuous functions $f\colon\mathbb{R}\to\mathbb{R}$ which posses finite or infinite derivative at every point, but that the derivative of $f$ is infinite on a dense $G_\delta$-set. The first example of such function was given by Pompeiu in [18]. More recent description of such functions can be found in [20, sec. 9.7] and [5]. These examples show that a perfect set in Theorem 2 should be nowhere sense. Thus we will use a measure theoretical approach, in which the measure theoretical tools will be present only implicitly or, as in case of Fact 5, given together with a simple proof.

We extract the proof of next theorem from the proof, presented in [8], of a Lebesgue theorem that every monotone function $f\colon\mathbb{R}\to\mathbb{R}$ is differentiable almost everywhere.

Our proof of Theorem 2 is based on the following 1932 result of Riesz [19], known as the rising sun lemma. For reader's convenience we include its short proof.

**Lemma 3.** *If $g$ is a continuous function from a non-trivial interval $[a,b]$ into $\mathbb{R}$, then the set $U=\{x\in[a,b)\colon g(x)<g(y)\text{ for some }y\in(x,b]\}$ is open in $[a,b)$ and $g(c)\le g(d)$ for every open connected component $(c,d)$ of $U$.*

PROOF. It is clear that $U$ is open in $[a,b)$. To see the other part, let $(c,d)$ be a component of $U$. By continuity of $g$, it is enough to prove that $g(p)\le g(d)$ for every $p\in(c,d)$. Assume by way of contradiction that $g(d)<g(p)$ for some $p\in(c,d)$ and let $x\in[p,b]$ be a point at which $g\restriction[p,b]$ achieves the maximum. Then $g(d)<g(p)\le g(x)$ and so we must have $x\in[p,d)\subset U$, as otherwise $d$ would belong to $U$. But $x\in U$ contradicts the fact that $g(x)\ge g(y)$ for every $y\in(x,b]$. $\square$

**Remark 4.** In Lemma 3 we also have $g(c)\ge g(d)$, since $c\in[a,b)\setminus U$. But we do not actually need this fact.

For an interval $I$ let $\ell(I)$ be its length. We need the following simple well-known observations.

**Fact 5.** *Let $a < b$ and $\mathcal{J}$ be a family of open intervals with $\bigcup \mathcal{J} \subset (a,b)$.*

*(i) If $[\alpha,\beta] \subset \bigcup \mathcal{J}$, then $\sum_{I\in\mathcal{J}} \ell(I) > \beta - \alpha$.*

*(ii) If the intervals in $\mathcal{J}$ are pairwise disjoint, then $\sum_{I\in\mathcal{J}} \ell(I) \le b - a$.*

PROOF. (i) By compactness of $[\alpha,\beta]$ we can assume that $\mathcal{J}$ is finite, say of size $n$. Then (i) follows by an easy induction on $n$: if $(c,d) = J \in \mathcal{J}$ contains $\beta$, then either $c \le \alpha$, in which case (i) is obvious, or $\alpha < c$ and, by induction, $\sum_{I\in\mathcal{J}} \ell(I) = \ell(J) + \sum_{I\in\mathcal{J}\setminus\{J\}} \ell(I) > \ell([c,\beta]) + \ell([\alpha,c]) = \beta - \alpha$.

(ii) Once again, it is enough to show (ii) for finite $\mathcal{J}$, say of size $n$, by induction. Then, there is $(c,d) = J \in \mathcal{J}$ to the right of any $I \in \mathcal{J}\setminus\{J\}$. Hence, by induction, $\sum_{I\in\mathcal{J}} \ell(I) = \ell(J) + \sum_{I\in\mathcal{J}\setminus\{J\}} \ell(I) \le (b-c)+(c-a) = b-a$. $\square$

PROOF OF THEOREM 2. If there exists a nontrivial interval $[c,d] \subset [a,b]$ on which $f$ is constant, then clearly $P = [c,d]$ is as needed. So, we can assume that $f$ is strictly monotone on $[a,b]$. Also, replacing $f$ with $-f$, if necessary, we can also assume that $f$ is strictly increasing.

Fix $L > |q(a,b)| = \frac{f(b)-f(a)}{b-a}$ and define $g\colon \mathbb{R} \to \mathbb{R}$ as $g(t) = f(t) - Lt$. Then $g(a) = f(a) - La > f(b) - Lb = g(b)$. Let $m = \sup\{g(x)\colon x \in [a,b]\}$ and $\bar{a} = \sup\{x \in [a,b]\colon g(x) = m\}$. Then $f(\bar{a}) - L\bar{a} = g(\bar{a}) \ge g(a) > g(b) = f(b) - Lb$, so $a \le \bar{a} < b$ and we still have $L > |q(\bar{a},b)| = \frac{f(b)-f(\bar{a})}{b-\bar{a}}$. Moreover, $\bar{a}$ does not belong to the set

$$U = \{x \in [\bar{a},b)\colon g(y) > g(x) \text{ for some } y \in (x,b]\}$$

from Lemma 3 applied to $g$ on $[\bar{a},b]$. In particular, $U$ is open in $\mathbb{R}$ and the family $\mathcal{J}$ of all connected components of $U$ contains only open intervals $(c,d)$ for which, by Lemma 3, $g(c) \le g(d)$.

The set $P = [\bar{a},b] \setminus U \subset [a,b]$ is closed and for any $x < y$ in $P$ we have $f(y) - Ly = g(y) \le g(x) = f(x) - Lx$, that is, $|f(y) - f(x)| = f(y) - f(x) \le Ly - Lx = L|y-x|$. In particular, $f$ is Lipschitz on $P$ with constant $L$. It is enough to show that $P$ is uncountable.

To see this notice that for every $J = (c,d) \in \mathcal{J}$ we have $f(d) - Ld = g(d) \ge g(c) = f(c) - Lc$, that is, $\ell(f[J]) = f(d) - f(c) \ge L(d-c) = L\ell(J)$. Since the intervals in the family $\mathcal{J}^* = \{f[J]\colon \mathcal{J} \in \mathcal{J}\}$ are pairwise disjoint and contained in the interval $(f(\bar{a}), f(b))$, by Fact 5(ii) we have $\sum_{J^*\in\mathcal{J}^*} \ell(J^*) \le f(b) - f(\bar{a})$. So, $\sum_{J\in\mathcal{J}} \ell(J) \le \frac{1}{L}\sum_{J\in\mathcal{J}} \ell(f[J]) = \frac{1}{L}\sum_{J^*\in\mathcal{J}^*} \ell(J^*) \le \frac{f(b)-f(\bar{a})}{L} < b - \bar{a}$. Thus, by Fact 5(i), $P = [\bar{a},b] \setminus U = [\bar{a},b] \setminus \bigcup\mathcal{J} \ne \emptyset$. However, we need more, that $P$ cannot be contained in any countable set, say $\{x_n : n \in \mathbb{N}\}$. To see this, fix $\delta > 0$ such that $\frac{f(b)-f(\bar{a})}{L} + \delta < b - \bar{a}$, for every $n \in \mathbb{N}$ choose an interval $(c_n, d_n) \ni x_n$ of length $2^{-n}\delta$, and put $\hat{\mathcal{J}} = \mathcal{J} \cup \{(c_n, d_n) : n < \omega\}$. Then

$$\sum_{J \in \hat{\mathcal{J}}} \ell(J) = \sum_{J \in \mathcal{J}} \ell(J) + \sum_{n \in \mathbb{N}} \ell((c_n, d_n)) \leq \frac{f(b) - f(\bar{a})}{L} + \delta < \beta - \alpha$$

so, by Fact 5(i), $U \cup \bigcup_{n \in \mathbb{N}} (c_n, d_n) \supset U \cup \{x_n : n \in \mathbb{N}\}$ does not contain $[\bar{a}, b]$. In other words, $P = [\bar{a}, b] \setminus U$ is uncountable, as needed. □

**Remark 6.** A presented proof of Theorem 2 actually gives a stronger result, that the set $[a, b] \setminus P$ can have arbitrary small Lebesgue measure.

## 3 Perfect set on which the difference quotient map is uniformly continuous

The next proposition is a version of a theorem of Morayne [15], which implies that the conclusion of Proposition 7 holds when $f$, defined on a perfect subset of $\mathbb{R}$, is Lipschitz (i.e., the quotient map for such $f$ has bounded range). The key innovation in Proposition 7 is that we prove this result without assuming that $f$, or some restriction of it, is Lipschitz.

**Proposition 7.** *For every continuous $f : \mathbb{R} \to \mathbb{R}$ there exists a perfect set $Q \subset \mathbb{R}$ such that the quotient map $q \restriction Q^2$ is bounded and uniformly continuous.*

PROOF. If $f$ is monotone on some non-trivial interval $[a, b]$, then, by Theorem 2, there exists a perfect set $P \subset \mathbb{R}$ such that $f \restriction P$ is Lipschitz. Thus, by Morayne's theorem applied to $f \restriction P$, there exists a perfect $Q \subset P$ for which the quotient map $q$ is as needed. On the other hand, if $f$ is monotone on no non-trivial interval, then, by a 1953 theorem of Padmavally [17] (compare also [14, 13, 9]) there exists a perfect set $Q \subset \mathbb{R}$ on which $f$ is constant. Of course, the quotient map on such $Q$ is as desired. □

## 4 The main result

The following theorem is a restatement of Theorem 1 in a slightly different language.

**Theorem 8.** *For every continuous function $f : \mathbb{R} \to \mathbb{R}$ there exists a perfect set $Q \subset \mathbb{R}$ such that $f \restriction Q$ can be extended to $C^1$ function $F : \mathbb{R} \to \mathbb{R}$.*

Let $Q \subset \mathbb{R}$ be as Proposition 7. It is well known, see e.g. [12], that uniform continuity of $q \restriction Q^2$ implies that the assumptions of the Whitney's $C^1$ extension theorem (see [24]) are satisfied, that is, $f \restriction Q$ has a desired $C^1$ extension $F \colon \mathbb{R} \to \mathbb{R}$. The problem with the citation [12], and many other papers containing needed extension result, is that the proofs presented there can hardly be considered simple. Thus, we like conclude the extendability of $f \restriction Q$, having uniformly continuous $q \restriction Q^2$, to $C^1$ extension $F \colon \mathbb{R} \to \mathbb{R}$ from the following recent result of Ciesielska and Ciesielski [4] which has simple elementary proof.

For a bounded open interval $J$ let $I_J$ be the closed middle third of $J$ and for a perfect set $Q \subset \mathbb{R}$ let

$$\hat{Q} = Q \cup \bigcup \{I_J \colon J \text{ is a bounded connected component of } \mathbb{R} \setminus Q\}.$$

**Proposition 9.** [4] *Let $f \colon Q \to \mathbb{R}$, where $Q$ is a perfect subset of $\mathbb{R}$, and put $\hat{f} = \bar{f} \restriction \hat{Q}$, where $\bar{f} \colon \mathbb{R} \to \mathbb{R}$ is a linear interpolation of $f \restriction Q$. If $f \restriction Q$ is differentiable, then there exists a differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$ of $\hat{f}$. Moreover, $F$ is $C^1$ if, and only if, $\hat{f}$ is continuously differentiable.*

PROOF OF THEOREM 8. If $Q \subset \mathbb{R}$ is from Proposition 7, then $q \restriction Q^2$, defined on $Q^2 \setminus \Delta$, can be extended to uniformly continuous $\bar{q}$ on $Q^2$ and $f \colon Q \to \mathbb{R}$ is continuously differentiable with $(f \restriction Q)'(x) = \bar{q}(x, x)$ for every $x \in Q$. By Proposition 9, $\hat{f}$ is differentiable (as a restriction of differentiable $F$). In particular, $\hat{f}'(x) = F'(x) = (f \restriction Q)'(x)$ for every $x \in Q$ and $\hat{f}'(x) = \bar{q}(c, d)$ whenever $x \in I_J$, where $J = (c, d)$ is a bounded connected component of $\mathbb{R} \setminus Q$.

By Proposition 9, we need to show that $\hat{f}'$ is continuous. Clearly $\hat{f}'$ is continuous on $\hat{Q} \setminus Q$, as it is locally constant on this set. So, let $x \in Q$ and let $\varepsilon > 0$. We need to find an open $U$ containing $x$ such that $|\hat{f}'(x) - \hat{f}'(y)| < \varepsilon$ whenever $y \in \hat{Q} \cap U$. Since $\bar{q}$ is continuous, there exists an open $V \in \mathbb{R}^2$ containing $\langle x, x \rangle$ such that $|\hat{f}'(x) - \bar{q}(y, z)| = |\bar{q}(x, x) - \bar{q}(y, z)| < \varepsilon$ whenever $\langle y, z \rangle \in Q^2 \cap V$. Let $U_0$ be open interval containing $x$ such that $U_0^2 \subset V$ and let $U \subset U_0$ be an open set containing $x$ such that: if $U \cap I_J \neq \emptyset$ for some bounded connected component $J = (c, d)$ of $\mathbb{R} \setminus Q$, then $c, d \in U_0$. We claim that $U$ is as needed. Indeed, let $y \in \hat{Q} \cap U$. If $y \in Q$, then $\langle y, y \rangle \in U^2 \subset V$ and $|\hat{f}'(x) - \hat{f}'(y)| = |\bar{q}(x, x) - \bar{q}(y, y)| < \varepsilon$. Also, if $y \in I_J$ for some bounded connected component $J = (c, d)$ of $\mathbb{R} \setminus Q$, then $\langle c, d \rangle \in U_0^2 \subset V$ and, once again, $|\hat{f}'(x) - \hat{f}'(y)| = |\bar{q}(x, x) - \bar{q}(c, d)| < \varepsilon$. $\square$

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