Lipschitz Restrictions of Continuous Functions and a Simple Construction of Ulam-Zahorski C1 Interpolation

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LIPSCHITZ RESTRICTIONS OF CONTINUOUS FUNCTIONS AND A SIMPLE CONSTRUCTION OF ULAM-ZAHORSKI $C^1$ INTERPOLATION

Abstract

We present a simple argument that for every continuous function $f: \mathbb{R} \to \mathbb{R}$ its restriction to some perfect set is Lipschitz. We will use this result to provide an elementary proof of the $C^1$ free interpolation theorem, that for every continuous function $f: \mathbb{R} \to \mathbb{R}$ there exists a continuously differentiable function $g: \mathbb{R} \to \mathbb{R}$ which agrees with $f$ on an uncountable set. The key novelty of our presentation is that no part of it, including the cited results, requires from the reader any prior familiarity with the Lebesgue measure theory.

1 Introduction and background

The main result we like to discuss here is the following 1985 theorem of Agronsky, Bruckner, Laczkovich, and Preiss [1]. It implies that every continuous function $f: \mathbb{R} \to \mathbb{R}$ must have some traces of differentiability, even though there exist continuous functions $f: \mathbb{R} \to \mathbb{R}$ that are nowhere differentiable (see e.g. [10, 22, 23]) or, even stronger, nowhere approximately and $I$-approximately differentiable. In fact, the first coordinate of the classical Peano curve (i.e., $f_1: [0, 1] \to [0, 1]$, where $f = (f_1, f_2): [0, 1] \to [0, 1]^2$ is a continuous surjection constructed by Peano) has these properties, see [6] or
Such a function cannot agree with a $C^1$ function on a set which is either of second category or of positive Lebesgue measure.

**Theorem 1.** For every continuous $f : \mathbb{R} \to \mathbb{R}$ there is a continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that the set $[f = g] = \{ x \in \mathbb{R} : f(x) = g(x) \}$ is uncountable. In particular, $[f = g]$ contains a perfect set $P$ and the restriction $f \restriction P$ is continuously differentiable.

In the statement of Theorem 1 the differentiability of $h = f \restriction P$ is understood as the existence of its derivative, that is, of the function $h' : P \to \mathbb{R}$ defined, for every $p \in P$, as $h'(p) = \lim_{x \to p, x \in P} \frac{h(x) - h(p)}{x - p}$.

The story behind Theorem 1 spreads over a big part of the 20th century and is described in detail in [2] and [16]. Briefly, around 1940 S. Ulam asked, in Scottish Book, Problem 17.1, see [21], whether every continuous $f : \mathbb{R} \to \mathbb{R}$ agrees with some real analytic function on an uncountable set. Z. Zahorski showed, in his 1948 paper [25], that the answer is no: there exists a $C^\infty$ (i.e., infinitely many times differentiable) function which can agree with every real analytic function on at most finite set of points. At the same paper Zahorski stated a problem, refereed to as Ulam-Zahorski problem: does every continuous $f : \mathbb{R} \to \mathbb{R}$ agrees with some $C^\infty$ (or possibly $C^n$ or $D^n$) function on some uncountable set? Clearly, Theorem 1 shows that Ulam-Zahorski problem has an affirmative answer for the $C^1$ class of functions. This is the best possible result in this direction, since A. Olevskii constructed, in his 1994 paper [16], a continuous function which can agree with every $C^2$ function on at most countable set of points.

The format of our proof of Theorem 1 is relatively straightforward. First we provide a simple argument that for every continuous function $f : \mathbb{R} \to \mathbb{R}$ its restriction to some perfect set $P \subset \mathbb{R}$ is Lipschitz. Here the key case, presented in Sec. 2, is when $f$ is monotone. Then we will follow an argument of Morayne [15] to show that there is a perfect $Q \subset P$ for which $f \restriction Q$ satisfies the assumptions of Whitney’s $C^1$ extension theorem [24]. At this point, to make the argument more accessible, we point the reader to a version of Whitney’s $C^1$ extension theorem from [4], whose proof is elementary and simple.

\[^1\]Of course this result follows immediately from Theorem 1, as $g$ from Theorem 1 is Lipschitz on any bounded interval. However, we are after a simpler proof of Theorem 1, so using it to argue for our step to prove it is pointless.
2 Lipschitz restrictions of monotone continuous maps

In what follows \( f \) will always be a continuous function from \( \mathbb{R} \) into \( \mathbb{R} \), \( \Delta \) will stand for the set \( \{ (x,x) : x \in \mathbb{R} \} \), and \( q : \mathbb{R}^2 \setminus \Delta \to \mathbb{R} \) be the quotient function for \( f \), that is, defined as \( q(x,y) = \frac{f(x) - f(y)}{x-y} \). For \( Q \subset \mathbb{R} \) we will use the symbol \( q \restriction Q^2 \) to denote the restriction of \( q \) to the set \( Q^2 \setminus \Delta \).

**Theorem 2.** Assume that \( f : \mathbb{R} \to \mathbb{R} \) is monotone and continuous on a non-trivial interval \([a,b]\). For every \( L > |q(a,b)| \) there exists a closed uncountable set \( P \subset [a,b] \) such that \( f \restriction P \) is Lipschitz with constant \( L \).

The difficulty in proving Theorem 2 without measure theoretical tools comes from the fact that there exist strictly increasing continuous functions \( f : \mathbb{R} \to \mathbb{R} \) which possess finite or infinite derivative at every point, but that the derivative of \( f \) is infinite on a dense \( G_\delta \)-set. The first example of such function was given by Pompeiu in [18]. More recent description of such functions can be found in [20, sec. 9.7] and [5]. These examples show that a perfect set in Theorem 2 should be nowhere sense. Thus we will use a measure theoretical approach, in which the measure theoretical tools will be present only implicitly or, as in case of Fact 5, given together with a simple proof.

We extract the proof of next theorem from the proof, presented in [8], of a Lebesgue theorem that every monotone function \( f : \mathbb{R} \to \mathbb{R} \) is differentiable almost everywhere.

Our proof of Theorem 2 is based on the following 1932 result of Riesz [19], known as the rising sun lemma. For reader’s convenience we include its short proof.

**Lemma 3.** If \( g \) is a continuous function from a non-trivial interval \([a,b]\) into \( \mathbb{R} \), then the set \( U = \{ x \in [a,b) : g(x) < g(y) \text{ for some } y \in (x,b) \} \) is open in \([a,b)\) and \( g(c) \leq g(d) \) for every open connected component \((c,d)\) of \( U \).

**Proof.** It is clear that \( U \) is open in \([a,b)\). To see the other part, let \((c,d)\) be a component of \( U \). By continuity of \( g \), it is enough to prove that \( g(p) \leq g(d) \) for every \( p \in (c,d) \). Assume by way of contradiction that \( g(d) < g(p) \) for some \( p \in (c,d) \) and let \( x \in [p,b) \) be a point at which \( g \restriction [p,b) \) achieves the maximum. Then \( g(d) < g(p) \leq g(x) \) and so we must have \( x \in [p,d) \subset U \), as otherwise \( d \) would belong to \( U \). But \( x \in U \) contradicts the fact that \( g(x) \geq g(y) \) for every \( y \in (x,b) \).

**Remark 4.** In Lemma 3 we also have \( g(c) \geq g(d) \), since \( c \in [a,b) \setminus U \). But we do not actually need this fact.
For an interval $I$ let $\ell(I)$ be its length. We need the following simple well-known observations.

**Fact 5.** Let $a < b$ and $\mathcal{J}$ be a family of open intervals with $\bigcup \mathcal{J} \subset (a, b)$.

(i) If $[\alpha, \beta] \subset \bigcup \mathcal{J}$, then $\sum_{I \in \mathcal{J}} \ell(I) > \beta - \alpha$.

(ii) If the intervals in $\mathcal{J}$ are pairwise disjoint, then $\sum_{I \in \mathcal{J}} \ell(I) \leq b - a$.

**Proof.** (i) By compactness of $[\alpha, \beta]$ we can assume that $\mathcal{J}$ is finite, say of size $n$. Then (i) follows by an easy induction on $n$: if $(c, d) = J \in \mathcal{J}$ contains $\beta$, then either $c \leq \alpha$, in which case (i) is obvious, or $\alpha < c$ and, by induction, $\sum_{I \in \mathcal{J}} \ell(I) = \ell(J) + \sum_{I \in \mathcal{J} \setminus \{J\}} \ell(I) > \ell([\alpha, \beta]) + \ell([\alpha, c]) = \beta - \alpha$.

(ii) Once again, it is enough to show (ii) for finite $\mathcal{J}$, say of size $n$, by induction. Then, there is $(c, d) = J \in \mathcal{J}$ to the right of any $I \in \mathcal{J} \setminus \{J\}$. Hence, by induction, $\sum_{I \in \mathcal{J}} \ell(I) = \ell(J) + \sum_{I \in \mathcal{J} \setminus \{J\}} \ell(I) \leq (b - c) + (c - a) = b - a$. \qed

**Proof of Theorem 2.** If there exists a nontrivial interval $[c, d] \subset [a, b]$ on which $f$ is constant, then clearly $P = [c, d]$ is as needed. So, we can assume that $f$ is strictly monotone on $[a, b]$. Also, replacing $f$ with $-f$, if necessary, we can also assume that $f$ is strictly increasing.

Fix $L > |g(a, b)| = \frac{f(b) - f(a)}{b - a}$ and define $g: \mathbb{R} \to \mathbb{R}$ as $g(t) = f(t) - Lt$. Then $g(a) = f(a) - La > f(b) - Lb = g(b)$. Let $m = \sup\{g(x): x \in [a, b]\}$ and $\bar{a} = \sup\{x \in [a, b]: g(x) > m\}$. Then $f(\bar{a}) - L\bar{a} = g(\bar{a}) \geq g(a) > g(b) = f(b) - Lb$, so $a \leq \bar{a} < b$ and we still have $L > |q(\bar{a}, b)| = \frac{f(b) - f(\bar{a})}{b - \bar{a}}$. Moreover, $\bar{a}$ does not belong to the set

$$U = \{x \in [\bar{a}, b): g(y) > g(x) \text{ for some } y \in (x, b]\}$$

from Lemma 3 applied to $g$ on $[\bar{a}, b]$. In particular, $U$ is open in $\mathbb{R}$ and the family $\mathcal{J}'$ of all connected components of $U$ contains only open intervals $(c, d)$ for which, by Lemma 3, $g(c) \leq g(d)$.

The set $P = [\bar{a}, b] \setminus U \subset [a, b]$ is closed and for any $x < y$ in $P$ we have $f(y) - Ly = g(y) \leq g(x) = f(x) - Lx$, that is, $|f(y) - f(x)| = f(y) - f(x) \leq Ly - Lx = L|y - x|$. In particular, $f$ is Lipschitz on $P$ with constant $L$. It is enough to show that $P$ is uncountable.

To see this notice that for every $J = (c, d) \in \mathcal{J}$ we have $f(d) - Ld = g(d) \geq g(c) = f(c) - Lc$, that is, $\ell(f[J]) = f(d) - f(c) \geq L(d - c) = L\ell(J)$. Since the intervals in the family $\mathcal{J}' = \{f[J]: J \in \mathcal{J}\}$ are pairwise disjoint and contained in the interval $(f(\bar{a}), f(b))$, by Fact 5(ii) we have $\sum_{J \in \mathcal{J}'} \ell(J^*) \leq \ell(f(\bar{a}) - f(b)).$ So, $\sum_{J \in \mathcal{J}} \ell(J) \leq \sum_{J \in \mathcal{J}} \ell(f[J]) = \sum_{J' \in \mathcal{J}'} \ell(J^*) \leq \frac{L(b - \bar{a})}{b - a} < b - \bar{a}$.

Thus, by Fact 5(i), $P = [\bar{a}, b] \setminus U = [\bar{a}, b] \setminus \bigcup \mathcal{J} \neq \emptyset$. However, we need more,
that \( P \) cannot be contained in any countable set, say \( \{x_n: n \in \mathbb{N}\} \). To see this, fix \( \delta > 0 \) such that \( \frac{f(b) - f(\bar{a})}{L} + \delta < b - \bar{a} \), for every \( n \in \mathbb{N} \) choose an interval \((c_n, d_n) \ni x_n \) of length \( 2^{-n}\delta \), and put \( \mathcal{J} = \mathcal{J} \cup \{(c_n, d_n): n < \omega\} \). Then

\[
\sum_{J \in \mathcal{J}} \ell(J) = \sum_{J \in \mathcal{J}} \ell(J) + \sum_{n \in \mathbb{N}} \ell((c_n, d_n)) \leq \frac{f(b) - f(\bar{a})}{L} + \delta < \beta - \alpha
\]

so, by Fact 5(i), \( U \cup \bigcup_{n \in \mathbb{N}} (c_n, d_n) \supset U \cup \{x_n: n \in \mathbb{N}\} \) does not contain \([\bar{a}, b]\).

In other words, \( P = [\bar{a}, b] \setminus U \) is uncountable, as needed.

**Remark 6.** A presented proof of Theorem 2 actually gives a stronger result, that the set \([a, b] \setminus P\) can have arbitrary small Lebesgue measure.

### 3 Perfect set on which the difference quotient map is uniformly continuous

The next proposition is a version of a theorem of Morayne [15], which implies that the conclusion of Proposition 7 holds when \( f \), defined on a perfect subset of \( \mathbb{R} \), is Lipschitz (i.e., the quotient map for such \( f \) has bounded range). The key innovation in Proposition 7 is that we prove this result without assuming that \( f \), or some restriction of it, is Lipschitz.

**Proposition 7.** For every continuous \( f: \mathbb{R} \to \mathbb{R} \) there exists a perfect set \( Q \subset \mathbb{R} \) such that the quotient map \( q \mid Q^2 \) is bounded and uniformly continuous.

**Proof.** If \( f \) is monotone on some non-trivial interval \([a, b]\), then, by Theorem 2, there exists a perfect set \( P \subset \mathbb{R} \) such that \( f \mid P \) is Lipschitz. Thus, by Morayne’s theorem applied to \( f \mid P \), there exists a perfect \( Q \subset P \) for which the quotient map \( q \) is as needed. On the other hand, if \( f \) is monotone on no non-trivial interval, then, by a 1953 theorem of Padmavally [17] (compare also [14, 13, 9]) there exists a perfect set \( Q \subset \mathbb{R} \) on which \( f \) is constant. Of course, the quotient map on such \( Q \) is as desired.

### 4 The main result

The following theorem is a restatement of Theorem 1 in a slightly different language.

**Theorem 8.** For every continuous function \( f: \mathbb{R} \to \mathbb{R} \) there exists a perfect set \( Q \subset \mathbb{R} \) such that \( f \mid Q \) can be extended to \( C^1 \) function \( F: \mathbb{R} \to \mathbb{R} \).
Let $Q \subset \mathbb{R}$ be as Proposition 7. It is well known, see e.g. [12], that uniform continuity of $q | Q^2$ implies that the assumptions of the Whitney’s $C^1$ extension theorem (see [24]) are satisfied, that is, $f | Q$ has a desired $C^1$ extension $F: \mathbb{R} \to \mathbb{R}$. The problem with the citation [12], and many other papers containing needed extension result, is that the proofs presented there can hardly be considered simple. Thus, we like conclude the extendability of $f | Q$, having uniformly continuous $q | Q^2$, to $C^1$ extension $F: \mathbb{R} \to \mathbb{R}$ from the following recent result of Ciesielska and Ciesielski [4] which has simple elementary proof.

For a bounded open interval $J$ let $I_J$ be the closed middle third of $J$ and for a perfect set $Q \subset \mathbb{R}$ let

$$\check{Q} = Q \cup \bigcup \{I_J : J \text{ is a bounded connected component of } \mathbb{R} \setminus Q\}.$$ 

**Proposition 9.** [4] Let $f: Q \to \mathbb{R}$, where $Q$ is a perfect subset of $\mathbb{R}$, and put $\hat{f} = f | \check{Q}$, where $\hat{f}: \mathbb{R} \to \mathbb{R}$ is a linear interpolation of $f | Q$. If $f | Q$ is differentiable, then there exists a differentiable extension $F: \mathbb{R} \to \mathbb{R}$ of $\hat{f}$. Moreover, $F$ is $C^1$ if, and only if, $\hat{f}$ is continuously differentiable.

**Proof of Theorem 8.** If $Q \subset \mathbb{R}$ is from Proposition 7, then $q | Q^2$, defined on $Q^2 \setminus \Delta$, can be extended to uniformly continuous $\tilde{q}$ on $Q^2$ and $f: Q \to \mathbb{R}$ is continuously differentiable with $(f | Q)'(x) = \tilde{q}(x, x)$ for every $x \in Q$. By Proposition 9, $\hat{f}$ is differentiable (as a restriction of differentiable $F$). In particular, $\hat{f}'(x) = F'(x) = (f | Q)'(x)$ for every $x \in Q$ and $\hat{f}'(x) = \tilde{q}(c, d)$ whenever $x \in I_J$, where $J = (c, d)$ is a bounded connected component of $\mathbb{R} \setminus Q$.

By Proposition 9, we need to show that $\hat{f}'$ is continuous. Clearly $\hat{f}'$ is continuous on $Q \setminus Q$, as it is locally constant on this set. So, let $x \in Q$ and let $\varepsilon > 0$. We need to find an open $U$ containing $x$ such that $|\hat{f}'(x) - \hat{f}'(y)| < \varepsilon$ whenever $y \in Q \cap U$. Since $\tilde{q}$ is continuous, there exists an open $V \in \mathbb{R}^2$ containing $(x, x)$ such that $|\tilde{q}'(x) - \tilde{q}'(y, z)| = |\tilde{q}(x, x) - \tilde{q}(y, z)| < \varepsilon$ whenever $(y, z) \in Q^2 \cap V$. Let $U_0$ be open interval containing $x$ such that $U_0^2 \subset V$ and let $U \subset U_0$ be an open set containing $x$ such that: if $U \cap I_J \neq \emptyset$ for some bounded connected component $J = (c, d)$ of $\mathbb{R} \setminus Q$, then $c, d \in U_0$. We claim that $U$ as needed. Indeed, let $y \in Q \cap U$. If $y \in Q$, then $(y, y) \in U^2 \subset V$ and $|\hat{f}'(x) - \hat{f}'(y)| = |\tilde{q}(x, x) - \tilde{q}(y, y)| < \varepsilon$. Also, if $y \in I_J$ for some bounded connected component $J = (c, d)$ of $\mathbb{R} \setminus Q$, then $(c, d) \in U_0^2 \subset V$ and, once again, $|\hat{f}'(x) - \hat{f}'(y)| = |\tilde{q}(x, x) - \tilde{q}(c, d)| < \varepsilon$. \qed
References


[17] K. Padmavally, *On the roots of equation $f(x) = \xi$ where $f(x)$ is real and continuous in $(a,b)$, but monotonic in no subinterval of $(a,b)$*, Proc. Amer. Math. Soc., 4 (1953), 839–841.


