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SOME ADDITIVE DARBOUX–LIKE FUNCTIONS

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Abstract. In this note we will construct several additive Darboux-like functions $f\colon \mathbb{R} \to \mathbb{R}$ answering some problems from (an earlier version of) [4]. In particular, in Section 2 we will construct, under different additional set theoretical assumptions, additive almost continuous (in sense of Stallings) functions $f\colon \mathbb{R} \to \mathbb{R}$ whose graph is either meager or null in the plane. In Section 3 we will construct an additive almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ which has the Cantor intermediate value property but is discontinuous on any perfect set. In particular, such an f does not have the strong Cantor intermediate value property.

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1. Preliminaries

Our terminology is standard and follows [3]. We consider only real-valued functions of one real variable. No distinction is made between a function and its graph. By $\mathbb{R}$ and $\mathbb{Q}$ we denote the set of all real and rational numbers, respectively. We will consider $\mathbb{R}$ and $\mathbb{R}^2$ as linear spaces over $\mathbb{Q}$. In particular, for a subset X of either $\mathbb{R}$ or $\mathbb{R}^2$ we will use the symbol $\mathrm{LIN}_{\mathbb{Q}}(X)$ to denote the smallest linear subspace (of $\mathbb{R}$ or $\mathbb{R}^2$) over $\mathbb{Q}$ that contains X. Recall also that if $D \subset \mathbb{R}$ is linearly independent over $\mathbb{Q}$ and $f\colon D \to \mathbb{R}$ then $F = \mathrm{LIN}_{\mathbb{Q}}(f) \subset \mathbb{R}^2$ is an additive function (see definition below) from $\mathrm{LIN}_{\mathbb{Q}}(D)$ into $\mathbb{R}$. Any linear basis of $\mathbb{R}$ over $\mathbb{Q}$ will be referred as a *Hamel basis*. By a Cantor set we mean any nonempty perfect nowhere dense subset of $\mathbb{R}$.

The ordinal numbers will be identified with the sets of all their predecessors, and cardinals with the initial ordinals. In particular $2 = \{0, 1\}$, and the first infinite ordinal ω number is equal to the set of all natural numbers $\{0, 1, 2, \dots\}$. The family of all functions from a set X into Y is denoted by Y^X. In particular, 2^n will stand for the set of all sequences $s\colon \{0, 1, 2, \dots, n-1\} \to \{0, 1\}$, while $2^{<\omega} = \bigcup_{n<\omega} 2^n$ is the set of all finite sequences into 2. The symbol $|X|$ stands for the cardinality of a set X. The cardinality of $\mathbb{R}$, is denoted by $\mathfrak{c}$ and referred as *continuum*. A set $S \subset \mathbb{R}$ is said to be $\mathfrak{c}$-*dense* if $|S \cap (a, b)| = \mathfrak{c}$ for every $a < b$.

We will use also the following terminology [4]. A function $f\colon \mathbb{R} \to \mathbb{R}$ is:

- *additive* if $f(x + y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$;
- *almost continuous* (in sense of Stallings) if each open subset of $\mathbb{R} \times \mathbb{R}$ containing the graph of f contains also a continuous function from $\mathbb{R}$ to $\mathbb{R}$ [11];
- has the *Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f[C] \subset K$;
- has the *strong Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f[C] \subset K$ and $f \restriction C$ is continuous.

Recall also that if the graph of $f\colon \mathbb{R} \to \mathbb{R}$ intersects every closed subset B of $\mathbb{R}^2$ which projection $\mathrm{pr}(B)$ onto the x-axis has nonempty interior then f is almost continuous. (See e.g. [10].)

2. An additive discontinuous almost continuous function with a small graph

In this section we will show that the continuum hypothesis implies the existence of an additive almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ whose graph is first category (or null) in the plane. This answers a question of Grande [5]. (See also [6] and [4, Question 5.2].) The author likes here to thank Udayan B. Darji for very helpful conversations on the subject.

Theorem 2.1. *For $i = 1, 2$ let $S_i \subset \mathbb{R}$ be such that $q \cdot S_i \subset S_i$ for every $q \in \mathbb{Q}$ and that the set*

$$\bigcap_{r\in T}(r + S_i)$$

is $\mathfrak{c}$-dense for any subset T of $\mathbb{R}$ of cardinality less than continuum. Then there exists an additive discontinuous almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \subset (S_1 \times \mathbb{R}) \cup (\mathbb{R} \times S_2)$.

Before we prove this theorem we like to notice the following corollary.

Corollary 2.2.

(1) *If $\mathbb{R}$ is not a union of less than continuum meager sets then there exists an additive discontinuous almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with the graph of measure zero.*

(2) *If $\mathbb{R}$ is not a union of less than continuum sets of measure zero then there exists an additive discontinuous almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with a meager graph.*

Proof. (1) Let S be a dense G_δ subset of $\mathbb{R}$ of measure zero and put $S_1 = S_2 = \bigcup_{q\in\mathbb{Q}} q \cdot S$. Then the sets S_1 and S_2 satisfy the assumptions of Theorem 2.1, while the set $(S_1 \times \mathbb{R}) \cup (\mathbb{R} \times S_2)$ has measure zero.

(2) Replace S with a meager set of full measure. □

Proof of Theorem 2.1. Let $S = (S_1 \times \mathbb{R}) \cup (\mathbb{R} \times S_2)$, and $\{A, C\}$ be a partition of $\mathfrak{c}$ with each set having cardinality $\mathfrak{c}$. Let $\{B_\xi \colon \xi \in A\}$ be an enumeration of all closed subsets B of $\mathbb{R}^2$ with $\mathrm{pr}(B)$ having nonempty interior, and $\{r_\xi \colon \xi \in C\}$ be an enumeration of $\mathbb{R}$. By induction on $\xi < \mathfrak{c}$ we will choose a sequence $\langle\langle x_\xi, y_\xi\rangle \colon \xi < \mathfrak{c}\rangle$ such that the following inductive assumptions are satisfied for every $\xi < \mathfrak{c}$.

(i) $x_\xi \notin \mathrm{LIN}_\mathbb{Q}(\{x_\zeta \colon \zeta < \xi\})$.

(ii) $f_\xi = \mathrm{LIN}_\mathbb{Q}(\{\langle x_\zeta, y_\zeta\rangle \colon \zeta \le \xi\}) \subset S$.

(iii) If $\xi \in A$ then $\langle x_\xi, y_\xi\rangle \in B_\xi$.

(iv) If $\xi \in C$ then $r_\xi \in \mathrm{LIN}_\mathbb{Q}(\{x_\zeta \colon \zeta \le \xi\})$.

Note first if we have such a sequence then, by (i) and (iv) the set $\{x_\xi \colon \xi < \mathfrak{c}\}$ is a Hamel basis. Thus $f = \mathrm{LIN}_{\mathbb{Q}}(\{\langle x_\xi, y_\xi\rangle \colon \xi < \mathfrak{c}\})$ is an additive function from $\mathbb{R}$ into $\mathbb{R}$ for which, by (ii), $f \subset S = (S_1 \times \mathbb{R}) \cup (\mathbb{R} \times S_2)$. Moreover, by (iii), f is almost continuous and has a dense graph in $\mathbb{R}^2$.

To construct a sequence as described above assume that for some $\xi < \mathfrak{c}$ the sequence $\langle\langle x_\zeta, y_\zeta\rangle \colon \zeta < \xi\rangle$ satisfying (i)–(iv) is already constructed. Then, by the inductive hypothesis,

$$g_\xi = \mathrm{LIN}_{\mathbb{Q}}(\{\langle x_\zeta, y_\zeta\rangle \colon \zeta < \xi\}) = \bigcup_{\zeta<\xi} f_\zeta \subset S.$$

Let D_ξ be the domain of g_ξ. The difficulty in choosing $\langle x_\xi, y_\xi\rangle$ is to make sure that

$$f_\xi = \{\langle x, y\rangle + q \cdot \langle x_\xi, y_\xi\rangle \colon \langle x, y\rangle \in g_\xi \ \&\ q \in \mathbb{Q}\} \subset S$$

which is equivalent to the choice of $\langle x_\xi, y_\xi\rangle$ from the set

$$\bigcap_{\langle x,y\rangle \in g_\xi} [\langle x, y\rangle + S)] \supset \left[\left(\bigcap_{x\in D_\xi} (x + S_1)\right) \times \mathbb{R}\right] \cup \left[\mathbb{R} \times \left(\bigcap_{x\in D_\xi} (g_\xi(x) + S_2)\right)\right].$$

(Note that S is closed under rational multiplication.)

Assume first that $\xi \in C$. If $r_\xi \notin D_\xi$ put $x_\xi = r_\xi$. Otherwise pick an arbitrary $x_\xi \in \mathbb{R} \setminus D_\xi$. This will guarantee (i) and (iv). In order to have (ii) choose y_ξ from $\bigcap_{x\in D_\xi}(g_\xi(x) + S_2)$, which is nonempty by the assumption from the theorem since $|D_\xi| \le |\xi| + \omega < \mathfrak{c}$. Then $\langle x_\xi, y_\xi\rangle \in \mathbb{R} \times \left(\bigcap_{x\in D_\xi}(g_\xi(x) + S_2)\right)$ implying (ii).

To finish the proof, assume that $\xi \in A$. The set $T = \bigcap_{x\in D_\xi}(x + S_1)$ is $\mathfrak{c}$-dense so we can choose $x_\xi \in T \cap \mathrm{pr}(B_\xi) \setminus D_\xi$. Take y_ξ such that $\langle x_\xi, y_\xi\rangle \in B_\xi$. Then (i), (ii), and (iii) are satisfied. □

To state the last corollary of this section we need the following lemma, that seems to have an interest of its own.

Lemma 2.3. *There exists a meager set $S \subset \mathbb{R}$ of measure zero with the properties that $p + q \cdot S \subset S$ for every $p, q \in \mathbb{Q}$, and the set*

$$\bigcap_{i<\omega} (r_i + S)$$

contains a perfect set for every sequence $\langle r_i \in \mathbb{R} \colon i < \omega\rangle$.

Proof. For $1 < k < \omega$ and a sequence $\langle s_n \subset n\colon k \le n < \omega\rangle$ of nonempty sets let

$$T(\langle s_n\rangle) = \left\{\sum_{n=2}^{\infty} \frac{i_n}{n!}\colon \forall k \le n < \omega\ (i_n \in s_n)\right\}.$$

Note that $T(\langle s_n\rangle)$ is a nonempty closed subset of $[0,1]$. It is nowhere dense, unless $s_n = n$ for all but finitely many n. Moreover, if there exists $k \le N < \omega$ such that $s_n = n$ for all $n > N$ then

$$m(T(\langle s_n\rangle)) = \prod_{n=k}^{N} \frac{|s_n|}{n}.$$

Also if $c_n = n-1$ for $k \le n < \omega$ then we denote the set $T(\langle c_n\rangle)$ by T^k. It is easy to see that

$$m(T^k) = \prod_{n=k}^{\infty} \frac{n-1}{n} = 0 \ \text{ for every } k.$$

Define

$$S = \bigcup\left\{p + qT^k\colon p, q \in \mathbb{Q}\ \&\ 1 < k < \omega\right\}.$$

Then S is meager, has measure zero, and is closed under $\mathbb{Q}$ addition and multiplication. To finish the proof, choose a sequence $\langle r_i \in \mathbb{R}\colon i < \omega\rangle$. It is enough to prove that $\bigcap_{3<i<\omega}(r_i + S)$ contains a perfect set. To prove this notice first that for every $r \in \mathbb{R}$ and every $1 < k < \omega$ there exists a sequence $\langle s_n \subset n\colon k \le n < \omega\rangle$ with each $|s_n| \ge n-2$ and such that

$$T(\langle s_n\rangle) \subset r + \bigcup\left\{p + T^k\colon p \in \mathbb{Q}\right\} \subset r + S.$$

This follows from the fact that if $x, y \in T^k$ have the same "m-th digit" i_m in the representation $\sum_{n=2}^{\infty}(i_n/n!)$, then the "$m$-th digits" of $r+x$ and $r+y$ can differ by at most 1 modulo m. (To see it, assume that r is of the form $p + \sum_{n=2}^{\infty}(j_n/n!)$ with p being an integer. Then $x + p + \sum_{n=2}^{m}(j_n/n!)$ and $y + p + \sum_{n=2}^{m}(j_n/n!)$ have the same "m-th digit", while by adding to any number the reminder $\sum_{n=m+1}^{\infty}(j_n/n!)$ of r, we increase the "m-th digit" by either 0 or 1 modulo m.)

Now, for each $3 < i < \omega$ choose a sequence $\langle s_n^i \subset n\colon 2i \le n < \omega\rangle$ with $|s_n^i| \ge n-2$ for every $n \ge 2$ for which

$$T(\langle s_n^i\rangle) \subset r_i + \bigcup\left\{p + T^{2i}\colon p \in \mathbb{Q}\right\} \subset r_i + S.$$

For every $8 \le n < \omega$ let $s_n = \bigcap_{8\le 2i\le n} s_n^i \subset n$. Then each s_n has at least two elements and

$$T(\langle s_n\rangle) \subset \bigcap_{3<i<\omega} T(\langle s_n^i\rangle) \subset \bigcap_{3<i<\omega}(r_i + S).$$

This finishes the proof. □

Corollary 2.4. *If the continuum hypothesis holds then there exists an additive discontinuous almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with the graph which is simultaneously meager and of measure zero.*

Proof. Apply Theorem 2.1 to $S_1 = S_2 = S$, where S is from Lemma 2.3. □

Problem 2.1. *Is it possible to find in ZFC an example of additive discontinuous almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with small graph (in sense of measure, category, or both)?*

3. An additive almost continuous function with the Cantor intermediate value property which is discontinuous on any perfect set

In this section we will construct in ZFC an additive almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with the Cantor intermediate value property which is discontinuous on any perfect set. In particular, such a function does not have a strong Cantor intermediate value property. A similar example has been constructed by K. Banaszewski and T. Natkaniec [2]: they constructed an almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with the Cantor intermediate value property which is of Sierpiński–Zygmund type, i.e., is discontinuous on any set of cardinality continuum. However, they had to use an additional set theoretical assumption in their construction ($\mathbb{R}$ is not a union of less than continuum many meager sets) which is necessary, since there is a model of ZFC with no Darboux (so almost continuous) Sierpiński–Zygmund function [1]. The constructed example answers Question 3.11 from [4].

Theorem 3.1. *There exists an additive almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ which has the Cantor intermediate value property, but is not continuous on any perfect set. In particular, f does not have the strong Cantor intermediate value property.*

The proof of the theorem is based on the following two lemmas.

Lemma 3.2. *Every perfect set $P_0 \subset \mathbb{R}$ has a perfect subset $P \subset P_0$ which is linearly independent over $\mathbb{Q}$.*

Proof. This can be proved by a minor modification of the proof presented in [7, thm. 2, Ch. XI sec. 7] that there exists a perfect subset of $\mathbb{R}$ which is linearly independent over $\mathbb{Q}$. (See also [8, 9].) □

Lemma 3.3. *There exists a Hamel basis H which can be partitioned onto the sets $\{P_\xi\colon \xi\le\mathfrak{c}\}$ with the following properties.*

(1) *For every $\xi<\mathfrak{c}$ the set P_ξ is perfect.*
(2) *Every nonempty interval contains continuum many sets P_ξ and continuum many points from $P_\mathfrak{c}$.*

Proof. Let P be a perfect set which is linearly independent over $\mathbb{Q}$. (See Lwmma 3.2.) Let K be a proper perfect subset of P and $\{x_\xi\colon \xi\le\mathfrak{c}\}$ be an enumeration of $P\setminus K$. Then there is a sequence $\langle\langle p_\xi,q_\xi\rangle\in(\mathbb{Q}\setminus\{0\})^2\colon \xi<\mathfrak{c}\rangle$ such that the sets $P_\xi=p_\xi x_\xi+q_\xi K$ satisfy the first part of (2). They also clearly satisfy (1). Now, it is easy to extend $\bigcup_{\xi<\mathfrak{c}}P_\xi$ to a Hamel basis H such that $P_\mathfrak{c}=H\setminus\bigcup_{\xi<\mathfrak{c}}P_\xi$ is a $\mathfrak{c}$-dense. □

Proof of Theorem 3.1. Let $\langle\langle I_\alpha,K_\alpha\rangle\colon \alpha<\mathfrak{c}\rangle$ be a list of all pairs $\langle I,K\rangle$ such that I is a nonempty open interval and K is a perfect set. By (2) of Lemma 3.3 we can reenumerate sets P_ξ to have $P_\alpha\subset I_\alpha$ for every $\alpha<\mathfrak{c}$. We will construct function f to have $f[P_\alpha]\subset K_\alpha$. This will guarantee the Cantor intermediate property of f. Next, let $\{B_\xi\colon \xi<\mathfrak{c}\}$ be an enumeration of all closed subsets B of $\mathbb{R}^2$ with $\mathrm{pr}(B)$ having nonempty interior, $\{x_\xi\colon \xi<\mathfrak{c}\}$ be an enumeration of the Hamel basis H from Lemma 3.3, and $\{C_\xi\colon \xi<\mathfrak{c}\}$ be an enumeration of all perfect sets C in $\mathbb{R}$ such that C is linearly independent over $\mathbb{Q}$. By induction on $\xi<\mathfrak{c}$ construct a sequence of functions $\langle f_\xi\colon D_\xi\to\mathbb{R}\colon \xi<\mathfrak{c}\rangle$ such that the following inductive assumptions are satisfied for every $\xi<\mathfrak{c}$.

(i) $\{D_\zeta\colon \zeta\le\xi\}$ are countable pairwise disjoint subsets of H.
(ii) If $x\in D_\xi\cap P_\alpha$ for some $\alpha<\mathfrak{c}$ then $f_\xi(x)\in K_\alpha$.
(iii) There exists $z\in D_\xi$ such that $\langle z,f_\xi(z)\rangle\in B_\xi$.
(iv) If $F_\xi=\mathrm{LIN}_\mathbb{Q}\left(\bigcup_{\zeta\le\xi}f_\zeta\right)$ then $x_\xi\in\mathrm{dom}(F_\xi)$ and $F_\xi\restriction C_\xi$ is discontinuous.

To construct such a sequence assume that for some $\xi<\mathfrak{c}$ a sequence $\langle f_\zeta\colon \zeta<\xi\rangle$ satisfying (i)–(iv) is already constructed. Let $V=\mathrm{LIN}_\mathbb{Q}\left(\bigcup_{\zeta<\xi}D_\zeta\right)$, choose a perfect subset $Z\subset C_\xi\setminus V$, and a countable dense subset D of Z. Also, let $A=\bigcup_{z\in D}\mathrm{supp}(z)$, where $\mathrm{supp}(z)$ is the support of z, i.e., the smallest set $S\subset H$ for which $z\in\mathrm{LIN}_\mathbb{Q}(S)$. Then A is countable, so we can choose $z_\omega\in Z\setminus\mathrm{LIN}_\mathbb{Q}(A)$ and a sequence $\langle z_n\in D\colon n<\omega\rangle$ converging to z_ω. Then $\{z_n\colon n\le\omega\}\subset C_\xi\setminus V$. Moreover, if $H_\eta=\mathrm{supp}(z_\eta)$ for $\eta\le\omega$ then there exists $y\in H_\omega\setminus(V\cup\bigcup\{H_n\colon n<\omega\})$. Choose

$$z\in\mathrm{pr}(B_\xi)\cap P_\mathfrak{c}\setminus\left(V\cup\bigcup\{H_n\colon n\le\omega\}\right)$$

and define

$$D_\xi = \Big(\{x_\xi, z\} \cup \bigcup\{H_n : n \leq \omega\}\Big) \setminus V.$$

Function f_ξ is defined on D_ξ as follows. For $x \in D_\xi \setminus \{y, x_\xi, z\}$ we define $f_\xi(x)$ arbitrarily, making only sure that condition (ii) is satisfied. By now, F_ξ is already defined on $\bigcup\{H_n : n \leq \omega\} \setminus \{y\}$. Thus, the sequence $\langle F_\xi(z_n) : n < \omega\rangle$ is already determined. If it does not converge, define f_ξ on y arbitrarily, making sure that condition (ii) is satisfied. If it converges to a limit L, define $f_\xi(y)$ to force $F_\xi(z_\omega) \neq L$. This can be done even if $y \in P_\alpha$ for some $\alpha < \mathfrak{c}$ since we still have many choices (all elements from K_α) for the value of $f_\xi(y)$. Notice that such a choice implies that $F_\xi \restriction \{z_n : n \leq \omega\}$ is discontinuous, while $\{z_n : n \leq \omega\} \subset C_\xi$. Thus (iv) is satisfied. We finish the construction by choosing $f_\xi(z)$ such that (iii) is satisfied, and $f_\xi(x_\xi)$ in arbitrary way (subject to condition (ii)) if $x_\xi \in D_\xi$ and $f_\xi(x_\xi)$ is not defined so far. The construction is completed.

Now, define $f = \mathrm{LIN}_{\mathbb{Q}}\Big(\bigcup_{\xi<\mathfrak{c}} f_\xi\Big)$. Then $f : \mathbb{R} \to \mathbb{R}$ is additive, since $\bigcup_{\xi<\mathfrak{c}} D_\xi = H$. Clearly, by (iv), f is discontinuous on any perfect set since, by Lemma 3.2, every perfect set contains some C_ξ. Also, (iii) guarantees that f is almost continuous, while (ii) guarantees that f has the Cantor intermediate value property. □

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